NAVORD REPORT 2958

A REVIEW OF INVESTIGATIONS ON TURBULENT FLOW WITH HEAT TRANSFER AT SMOOTH WALLS

4 NOVEMBER 1953

U. S. NAVAL ORDNANCE LABORATORY
WHITE OAK, MARYLAND

NAVORD Report 2958

A REVIEW OF INVESTIGATIONS ON TURBULENT FLOW WITH HEAT TRANSFER AT SMOOTH WALLS

Prepared by:

Phrixos J. Theodorides

ABSTRACT: This is a survey of recent developments about turbulent motion with heat transfer at smooth walls. Characteristics are discussed for such a flow in prismatic ducts with symmetric cross sections (closed or open) as well as in the boundary layer of a flat plate.

How does the ratio ($\beta = A_\tau/\mu$) of turbulent to molecular shear stress vary with a dimensionless distance ($\eta = yu^*/\nu$) from the wall? Of late, in reply to this question, the function $\beta(\eta)$ has been approximated analytically (References (m), (q), (t)) over several regions of the boundary layer. This allowed to trace as well, by integration, a mean value profile respectively for the velocity, and the temperature, directly as to the former, step-wise for the latter.

In the first part of this report, physical requirements of the lower atmosphere are transposed into analytic boundary conditions for the mean velocity profile. It is discussed in regard to a current formulation with two constants - power relation, logarithmic law - and also in respect to more recent endeavours (H. Reichardt, K. Elser) for an analytic expression of the continuity over the whole border film.

The early dichotomy (Prandtl, Taylor, etc.) of the boundary layer evolved through a subsequent trichotomy (von Kármán, Reichardt, etc.) to the latest theory (Reichardt, Elser, etc.). This is reflecting a continual decay of turbulence effects from the free stream to their extinction at the wall.

Reichardt's unified velocity profile (1951) is set forth and discussed as to the prevalent term, near the wall and in a median region respectively, and as to combining the terms in a complete profile.

U. S. NAVAL ORDNANCE LABORATORY
WHITE OAK, MARYLAND

For laminar flow with the above geometry, the temperature profile admits of a rigorous computation. Yet, for the thermal field of turbulence no general analytical solution is known.

The second part deals with progress toward solving the turbulent case by integrating the temperature equation numerically or graphically in successive steps. One or two steps appear sufficient for air and fluids with higher Prandtl-numbers (oils, etc.), while, at most, four steps would be needed in case of very low Pr-numbers (molten metals or alloys etc.).

The third part deals mainly with an iterative integration as a means of solving the heat transfer problem for turbulent flow. Two integrating varieties of the analytical method are discussed.

Theoretical results are compared with those of global, semi-empirical relations for the heat transfer coefficient.

The investigated results cover a wide range of the generalized Prandtl-number: $0 \leqslant Pr' \leqslant 10^3$

Numerical values of the heat transfer coefficient, as obtained by successive integration, appear, currently, in good agreement with those based on semi-empirical, product relations of powered similarity parameters as factors (References: (f_p), von Kármán, and (m_r), Reichardt, for high Pr-numbers; (q_p), Elser, and (s), Lyon, for low Pr-numbers).

The fourth part is presenting some concluding remarks.

NAVORD Report 2958 4 November 1953

This NAVORD is a review of recent progress on the influence of turbulence upon heat transfer at smooth walls for a flow in prismatic ducts or along a flat plate. Such developments concern problems currently encountered in the NOL AeroBallistic Research Department. The more recent of this reviewed research work was published in Göttingen and Zürich. No translation in English is available at present. Therefore, it is felt worth while to include and correlate the new methods and results in a comprehensive presentation.

This investigation was carried out under task number Re9a-108-1.

EDWARD L. WOODYARD
Captain, USN
Commander

H. H. KURZWEG, Chief
Aeroballistic Research Department
By direction

CONTENTS

Symbols

x, y, z . . . . Coordinates (x parallel to wall, y normal to wall, z normal to x-y plane)

$\bar{u} = t^{-1}\int_0^t u\,dt$ . . . . . Mean-temporal components of velocity in direction of coordinates x, y, z

$\bar{v}$, $\bar{w}$

u_f, v_f, w_f . . . . . . Instantaneous values of fluctuating parts of velocity components

$u = \bar{u} + u_f$ . . . . . Instantaneous values of resultant velocity components

$v = \bar{v} + v_f$

$w = \bar{w} + w_f$

$\left(\overline{u_f^2}\right)^{1/2} = \left(t^{-1}\int_0^t u_f^2\,dt\right)^{1/2}$ Root mean square (r m s) values (or effective values) of the fluctuating parts of velocity components. For fluctuations of sufficiently high frequency, a finite time interval "t" can be large enough for establishing quadratic mean values of the fluctuation, while the mean velocity may be assumed as practically constant.

$\left(\overline{v_f^2}\right)^{1/2}$, $\left(\overline{w_f^2}\right)^{1/2}$

A . . . . Cross-sectional area.

$u_m = A^{-1}\int_0^A \bar{u}\,dA$ . . . . Average value of mean velocity over cross-sectional area.

U . . . . Largest value of u-component of velocity.

T . . . . Absolute temperature.

$T - T_o$ . . . . Difference between temperature of moving fluid and wall temperature.

$\Theta = (T - T_o)_{max}$ . . Largest temperature difference.

$\vartheta = (T - T_o)\Theta^{-1}$ . . . Dimensionless temperature difference.

$T_m = A^{-1}\int_0^A \frac{u}{u_m} T\,dA$ . . Mean temperature of moving fluid over cross section.

ρ . . . . Density of moving fluid.

μ, $\nu = \mu\rho^{-1}$ . . Dynamic, and kinematic coefficient of molecular viscosity.

A_τ . . . . Coefficient of apparent viscosity due to turbulence.

A_q . . . . Coefficient of heat transport by convection through turbulence.

$m = \frac{A_q}{A_\tau}$ . . . Ratio between coefficients of turbulent heat flux density and apparent viscosity.

k . . . . Coefficient of molecular heat conductivity.

$i = \int c_p\,dT$ Enthalpy.

c_p . . . . Specific heat at constant pressure.

τ . . . . Shear stress.

q . . . . Heat flow density (heat per unit time and unit area).

$$\tau = \tau_t + \tau_m = (A_\tau + \mu)\frac{d\bar{u}}{dy}$$

$$q = q_t + q_m = (c_p A_q + k)\frac{d\bar{T}}{dy} = (A_q + k/c_p)\frac{d\bar{i}}{dy}$$

Subscript "t" refers to turbulent contribution, subscript "m" to molecular contribution.

$Pr = \frac{c_p \mu}{k}$ . . . . Prandtl-number

$$\left.\begin{array}{l} \beta = \dfrac{\tau_t}{\tau_m} = \dfrac{A_\tau}{\mu} \\[2ex] \dfrac{q_t}{q_m} = \dfrac{c_p A_q}{k} \end{array}\right\} \frac{q_t/q_m}{\tau_t/\tau_m} = \frac{A_q}{A_\tau}\frac{c_p\mu}{k} = \frac{A_q}{A_\tau} Pr = Pr'$$

Pr' . . . . Generalized Prandtl-number. It is a characteristic parameter of turbulent flow with heat transfer.

$\varphi = u U^{-1}$ . . . . . . Dimensionless velocity referred to free-stream value.

$\varphi_m = u_m U^{-1}$

$u^* = (\tau_0/\rho)^{1/2}$ . . . . Shear stress velocity at wall.

$\psi = u/u^*$ . . . . Dimensionless local velocity referred to shear velocity at wall.

$\eta = \frac{yu^*}{\nu}$ . . . . Dimensionless distance from wall.

$\zeta = \frac{zu^*}{\nu}$ . . . Dimensionless coordinate in z-direction.

$\eta_1 = (\eta)_{A_\tau = \mu}$ . . . Dimensionless width of viscous boundary defined as that distance from the wall beyond which turbulence accounts for a higher apparent shear stress than the part due to molecular agitation.

$\eta_2 = (\eta)_{A_q = k/c_p}$ . . . . Dimensionless width of the heat conducting boundary layer defined as that distance from the wall beyond which turbulence accounts for a higher transport of heat than the molecular conductivity does.

$a_\eta = \int_0^\eta \frac{d\eta}{(1+\beta)(1+\beta k')}$ Reichardt integrals concerning the temperature profile of the turbulent, heat transmitting boundary layer.

$a = \lim_{\eta \to \infty} a_\eta$ $\quad \lambda = \frac{q/q_0}{\tau/\tau_0} - 1$

$\bar{\lambda} = \int_0^1 \lambda \, c \quad$ for $\lambda < 1 \longrightarrow \lambda \ll 1$

along plate $\quad \lambda \approx 0$

$r = d/2$ . . . . Radius of circular tube or distance between wall and symmetry plane for a prismatic channel.

$\eta_{/c} = \frac{u^* r}{\nu}$ . . . . Dimensionless either radius or distance of wall from symmetry plane.

$y_c = r - y$ . . . . . Distance from axis of circular tube or from symmetry plane for a prismatic channel.

$Re = \frac{u_m d}{\nu}$ . . . . Reynolds number referred to the mean velocity over cross-section and to the tube diameter or largest channel dimension across the symmetry plane.

$Nu_{/m} = \frac{q_o d}{k(T_m - T_o)}$ Nusselt number referred to heat flow through the wall surface and to the average mean temperature of the moving fluid over a cross-section.

$Nu_e = \frac{q_o d}{k(T_e - T_o)}$ . . Nusselt number referred to the difference between the equilibrium temperature at an insulated wall and the temperature of the actual wall.

$h_u = \frac{q_o}{T_u - T_o}$ . . Dimensional heat transfer coefficient based on the temperature difference: $T_u - T_o$

$T_e = T + r_c \frac{V^2}{2c_p}$ . . Equilibrium temperature at a perfectly insulated wall.

$h' = \frac{q_o}{\rho c_p U \Theta}$ . . Dimensionless heat transfer coefficient based on largest velocity (U) and largest difference of temperature (Θ).

u_i . . . . . . Additional velocity for the median region of flow as introduced in new theory (Reichardt, 1950-51) in order to account for a slight variance between experimental results and the basic logarithmic law.

$\kappa, b, b_0, b_i, c_1, c$ Constant magnitudes.

$b_0 = \frac{1}{\kappa} \log \kappa + c_1$

A REVIEW OF INVESTIGATIONS ON TURBULENT FLOW WITH HEAT TRANSFER AT SMOOTH WALLS

Prefatory Remarks

1. The characteristics of turbulent flow in prismatic ducts with cross-sectional symmetry or along a flat plate are briefly surveyed. This is done with the aim of a closer approach to the heat transfer conditions at smooth walls. The wall temperature is assumed constant over the entire area of the solid boundary.

2. The duct may be either of closed cross section (tube) or open (channel).

3. The fields of mean velocity and mean enthalpy (or temperature) are based on a more complete account of the continuity of flow and the other physical requirements at the wall boundary and the free-stream portion of each profile.

4. The flow conditions are limited as a rule to negligible effects on the density of the fluid.

Part I. The Profile of Mean Velocity

Introduction

5. In this first part of the Report, only the velocity field is considered. After a brief historic survey, a discussion will follow on Reichardt's recent (1950-51) achievement in unifying the analytic expression for the complete velocity profile.

Boundary Conditions

6. A parallel mean flow in x-direction is considered. For the profile of mean velocity in the lower atmosphere, the physical requirements at a smooth wall are:

(α) Non-slip condition, i.e. zero value for the mean velocity:

$$\bar{u}_0 = 0$$

(β) Finite shear stress, i.e. finite gradient of mean velocity:

$$\left(\frac{d\bar{u}}{dy}\right)_0 \begin{matrix} \neq 0 \\ \neq \infty \end{matrix}$$

(γ) Continuity equation extended to include the three-dimensional group of the fluctuating parts (u_f, v_f, w_f) of the velocity components; for them:

$$\frac{\partial u_f}{\partial x} + \frac{\partial v_f}{\partial y} + \frac{\partial w_f}{\partial z} = 0$$

7. At the symmetry plane of a prismatical channel or at the axis of a circular tube, the physical requirements for the velocity profile are:

(δ) Vanishing velocity gradient: $\left(\frac{d\bar{u}}{dy}\right)_{center} = 0$

(ε) Finite curvature (Γ) at the transition point for two symmetrical branches of the velocity profile, i.e. an analytical maximum for the velocity at center:

$$\Gamma \approx \left(\frac{d^2u}{dy^2}\right)_{center} \begin{matrix} < 0 \\ \neq -\infty \end{matrix} \qquad (\bar{u})_{center} = u_{max}$$

Power Relation . Logarithmic Law

8. It is known that for the flow near a wall, though not too close to it, the measured velocity profile may be approximated by:

(a) A power relation with a fractional exponent: dependent on the Reynolds number:

$$\psi = \frac{u}{u^*} = c(n)\,\eta^{1/n(Re)} \qquad (1)$$

In a range of $40 < \eta < 60$, the Blasius values: $n = 7$, $c = 8.7$ are valid up to about $Re = 100{,}000$. Above this limit, the actual experiments account for increasing n-values with Re, viz. $n \rightarrow 8, 9, 10$.

(b) A logarithmic law with two constants:

$$\psi = \frac{u}{u^*} = b_0 + b_1 \ln \eta \qquad (2)$$

$b_0 = 5.5 \qquad b_1 = 2.5$ (Prandtl, Nikuradse, v. Kármán)

This formula would seem a more general approximation than the power law.

9. It is easy to check that the power relation (a) - though giving non-slip at the wall - violates there the boundary condition of a finite velocity gradient.

10. On the other hand, the logarithmic law, in form (b), violates at the wall the non-slip condition and that of a finite velocity gradient.

11. Besides, both laws (a) and (b) do not satisfy the center conditions for a symmetric duct or the circular tube.

12. In order to remedy the inconsistences in respect to the boundary conditions at the wall, it had become customary to assume that the turbulent agitation would not reach the fluid that moves in immediate vicinity of the wall. Thus, a purely laminar sublayer of very small thickness was postulated in a position between the wall and the turbulent region. Within the sublayer a linear velocity distribution was currently assumed, i.e.:

$$\left(\frac{u}{u^*}\right)_{sublayer} = \eta \qquad (3)$$

13. This implied a constant velocity gradient, and hence an invariable shear stress from the wall through the thickness of the sublayer. Yet, as a matter of experimental evidence, the shear stress:

$$\tau = \mu \frac{du}{dy}$$

diminishes with increasing distance from the wall. To account for this fact, formula (3) could of course be replaced by a parabolic branch (Poiseuille) to hold in the sublayer.

14. There are cogent considerations, however, that seem to preclude the very existence of a purely laminar sublayer. For a flow along a solid wall, it seems physically inconceivable to limit the spreading of turbulence to an arbitrary fluid boundary of a purely laminar sublayer. It would appear questionable as well to associate the fully developed turbulence to a local critical Reynolds number based on a reduced velocity within the boundary layer.

When the Reynolds number, referred to the mean ordinate of the velocity profile, becomes compatible with stability of turbulence, then this kind of flow is building up and spreading around so as to extend soon all over the cross section. Thus, finally a turbulent agitation becomes superposed over the molecular one throughout.

Continuity of Velocity Profile . Exponential Transition

15. In two different ways, Reichardt (1940) and Elser (1949) have expressed analytically the continuity of the velocity profile.

16. In Reference (m_α), an intermediate exponential law:

$$u/u^* = 15.5 - (13.5)\exp\left(\frac{2-\eta}{13.5}\right) \qquad (4)$$

is interpolated between the linear and the logarithmic region in a manner giving a continuous transition of slope at both ends of the intermediate zone.

17. In Reference (q_α) the continuity of the velocity profile is satisfied with another exponential function:

$$u/u^* = 15.95\left[1 - \exp\left(-\frac{\eta}{15.95}\right)\right] \qquad (5)$$

holding for: $0 \leq \eta \leq 46.26$

18. This exponential curve is starting directly from the wall (no laminar sublayer) and leading smoothly into the logarithmic branch at the other end.

19. However, neither of the above solutions seems to satisfy, in proximity of the wall, the requirement of continuity for the group of the fluctuating instantaneous components (u_f, v_f, w_f) that express the turbulent motion as superposed to the mean flow.

Reichardt's Unified Law for the Complete Velocity Profile

a. Boundary Layer and Generalized Prandtl-Number

20. More advanced investigations of recent date (Reichardt, Reference (m_6), 1950), have led to defining a characteristic thickness of the heat conducting boundary layer (η_2) as the dimensionless distance from the wall of an imaginary dividing line. At this distance there is equality between the molecular (k) and the turbulent ($c_p A_q$) coefficient of thermal conductivity. Among the two causes of heat conduction at a point with:

$\eta < \eta_2$: the molecular one prevails.

$\eta > \eta_2$: the turbulent one prevails.

21. Similarly, another characteristic thickness (η_1) is defined for the viscous boundary layer. It corresponds to an equality between molecular (μ) and turbulent (A_τ) coefficient of viscosity. Among the two causes of shear stress at a point with:

$\eta < \eta_1$: the molecular one prevails.

$\eta > \eta_1$: the turbulent one prevails.

22. These definitions presume a co-existence of molecular and turbulent effects. The preponderance of the former or the latter is depending on whether the dimensionless distance (η) from the wall is smaller or larger than the characteristic thickness of the respective boundary layer.

23. Considerations based on a thorough analysis of measurements by Reichardt and other investigators have shown that for a sufficiently large η, the molecular effects become negligible as compared to turbulence. Yet a converse assumption of a purely laminar sublayer at wall proximity is not free of contradictions.

24. The layer with prevalent molecular viscosity ($\mu > A_\tau$) seems to have a fairly constant thickness, say $\eta_1 = 11$ for $\mu = A_\tau$. This value of η_1, corresponds to: $\xi_0 = 5.5$ in the logarithmic profile (cf. formula (2), p. 9). However, the thickness (η_2) of the heat conducting layer appears strongly dependent on the general Prandtl number:

$$Pr' = (A_q/A_\tau)\, Pr$$

The actual picture is:

$$Pr' > 1 \longrightarrow \eta_2(Pr') < \eta_1$$
$$Pr' < 1 \longrightarrow \eta_2(Pr') > \eta_1$$

and experiments have shown that: $1 \leq A_q/A_\tau < 2$

25. As:
$$\frac{A_q}{A_\tau} = \frac{k}{c_p \mu} \, \frac{q_t/q_m}{\tau_t/\tau_m} \approx \frac{q_t/q_m}{\tau_t/\tau_m} \tag{6}$$

it follows that the relative influence of turbulence on heat conduction is, for actual conditions of $A_q/A_\tau > 1$, larger than its relative influence on diffusion of momentum (viscous effects). From good measurements, Reichardt has computed quite recently for a range of the general Prandtl number:
$0.72 \leq Pr' \leq 1000$, a corresponding range:

$12.5 \geq \eta_2 \geq 1.05$ for the thickness of the heat conducting boundary layer.

26. It follows that, at higher values of Pr', there is a large interval:

$$\Delta\eta = \eta_1 - (\eta_2)_{\text{large } Pr'} \longrightarrow \approx 10$$

(relatively: large minuend, small subtrahend) within which an assumed laminar sublayer would have to accommodate a prevalently turbulent heat flow with a purely laminar transport of momentum. This would be a basic contradiction, and the alternative of complicating the theory by implying a different mechanism of turbulence for the transport of heat than of momentum would not appear less objectionable. Hence the necessity of thinning down the sublayer so very much as to having it dropped completely.

b. Wall Region

27. Reichardt took in account the continuity requirement with respect to the three-dimensional group of the fluctuating parts (u_f, v_f, w_f) of the velocity components. Thus he obtained (1951) the following expression for the ratio (ρ) between turbulent and molecular shear stress in the vicinity of a wall:

$$\rho = \frac{A_\tau}{\mu} = \frac{1}{2\mu^{*2}} \overline{\left(\frac{\partial u_f}{\partial \eta}\right)_0 \frac{\partial}{\partial \zeta}\left(\frac{\partial \omega_f}{\partial \eta}\right)_0 \eta^3} \qquad (7)$$

28. In recent papers (References: (m_β), (m_γ)) it is assumed that, at the wall, the derivatives

$$\left(\frac{\partial u_f}{\partial \eta}\right)_0 \qquad \text{and} \qquad \frac{\partial}{\partial \zeta}\left(\frac{\partial \omega_f}{\partial \eta}\right)_0$$

are correlated to each other so as to yield a finite value for the average of their product:

$$\overline{\left(\frac{\partial u_f}{\partial \eta}\right)_0 \frac{\partial}{\partial \zeta}\left(\frac{\partial \omega_f}{\partial \eta}\right)_0}$$

29. It follows that the established boundary conditions of non-slip and finite shear stress at the wall have to be supplemented by added requirements. Thus, not only the function (A_τ/μ) itself, but its first and second derivative as well, with regard to η, have to vanish at the wall.

30. Rigorously, such a prevalence of the cubic term of a power series:

$$A_\tau/\mu = \sum (c_i \eta^i)$$

is well established only for a partly developed, two-dimensional state of turbulence close to the wall. Yet this is to hold, most likely, as well for the state of the fully developed, three-dimensional turbulence (Reference (m_γ)).

31. The integrated set of four boundary conditions at the wall and an assumed asymptotic trend of the function β toward the linear form $\kappa\eta$ at some distance from the wall constitute a solid physical background for the new Reichardt function (Figure 2):

$$\beta = \kappa\left(\eta - \eta_1 \tanh \eta/\eta_1\right) \tag{8}$$

32. For a flow with drop of pressure, the identity:

$$\frac{d\psi}{d\eta} = \frac{\tau_m}{\tau_o} \quad \text{with} \quad \psi = u/u^* \tag{9}$$

may be easily transformed into:

$$\frac{d\psi}{d\eta} = \frac{\tau/\tau_o}{1+\beta} \approx \frac{1}{1+\beta} \tag{10}$$

where the expression $1/1+\beta$ approximates fairly well a flow close to the wall of prismatic ducts. The cross section may be either closed (tube) or open (channel). This was obtained with a linear drop of shear stress from its value (τ_o) at the wall, and the neglect there of the term (y/r) as compared to unity.

33. The insertion of function (8) into (10) leads to the following differential equation:

$$d\psi = \frac{d\eta}{(1+\kappa\eta)\left(1 - \frac{\kappa\eta_1}{1+\kappa\eta}\tanh \eta/\eta_1\right)} \tag{10a}$$

34. Integration by parts yields:

$$\psi = \frac{1}{\kappa}\ln(1+\kappa\eta) + \int_0^{\eta} \frac{\kappa\eta_1 \tanh(\eta/\eta_1)\, d\eta}{(1+\kappa\eta)(1+\kappa\eta - \kappa\eta_1 \tanh \eta/\eta_1)} \tag{11}$$

35. Unfortunately, the integral (t) of the right-hand side turns out to be intractable. Approximate solutions have developed (Reference (m_β)) in two ways; these are:

a. Graphical integration and neglect of the unity against κy for high values of η.

b. Replacement of the intractable integrand by an approximate substitute such as to make the integration possible. From:

$$\frac{df}{d\eta} = \frac{\kappa\eta_1 \tanh(\eta/\eta_1)}{(1+\kappa\eta)(1+\kappa\eta-\kappa\eta_1\tanh\eta/\eta_1)} \approx \frac{c_1}{\eta}\left\{\exp(-\eta/\eta_1)+(b\eta-1)\exp(-b\eta)\right\} = \frac{df_1}{d\eta}$$

It follows that:

$$f_1 = c_1\left\{1-\exp(-\eta/\eta_1)-\frac{\eta}{\eta_1}\exp(-b\eta)\right\}$$

36. After carrying out the integration and determining the relevant constants from the boundary conditions, Reichardt obtained finally, for the wall region, the velocity profile:

$$\psi = u/u^* = \kappa^{-1}\ln(1+\kappa\eta) + c_1\left\{1-\exp(\eta/\eta_1)-(\eta/\eta_1)\exp(-b\eta)\right\} \quad (12)$$

with the constants (cf. p. 9 for ℓ_0, ℓ_1):

$$\kappa = 1/\ell_1 = 0.4$$
$$c_1 = \ell_0 - \frac{1}{\kappa}\ln\kappa \approx 7.79 \text{ for } \eta_1 \approx 11$$
$$b \approx 1/3$$

37. In Figure 3 the solution "a" (graphical integration) is marked by a full line whereas the dashed one represents solution "b" (approximate integration). Between "a" and "b", only insignificant discrepancies with alternating sign occur for certain regions of the dimensionless distance (η) from the wall.

In the same figure, the ratio τ_t/τ is plotted versus η ; the latter is ranging from 1 to 100.

c. Median Region

38. In the median region of a flow through prismatic ducts, the agitation is prevalently turbulent i.e. the effects of the molecular motion on the velocity profile may be neglected. The distance ($y_c = h-y$) from the median plane (channel), or the axis (tube), is a more convenient coordinate than y.

Therefore, the orgin of the ordinate in a direction normal to the wall is shifted from the wall to the center.

39. In Reference (m_y), the relevant velocity profile is based on accurate empirical data. The relative coefficient

$$A_\tau / \mu \eta_R$$

of the apparent shear stress was obtained by directly measuring the derivative of the stagnation pressure throughout the cross section with a double Pitot tube. Such experimental values obtained in a channel of rectangular cross section (Figures 4 and 5) were expressed by the formula:

$$\frac{A_\tau}{\mu \eta_R} = \frac{\kappa}{3}\left\{\frac{1}{2} + \left(\frac{y_c}{R}\right)^2\right\}\left\{1 - \left(\frac{y_c}{R}\right)^2\right\} \tag{13}$$

40. This relation is inserted in the counterpart of equation (3) which for the median region is:

$$d\varphi = - \frac{\tau/\tau_0}{A/\mu \eta_R}\, d\left(y_c/R\right)$$

41. After integrating and determining the relevant constant, the velocity profile becomes for nearly the entire cross section (except in vicinity to a wall):

$$\frac{u_m - u}{u^*} = \kappa^{-1} \ln\left(\frac{1 + 2(y_c/R)^2}{1 - (y_c/R)^2}\right) \tag{14}$$

where:

$$u_m = (u)_{y_c = 0}$$

42. In the graph (Figure 4) the measured relative velocity $\varphi = u/u_{max}$ as well as its derivative $d\varphi/dz$, and the computed second derivative $d^2\varphi/dz^2$ are all plotted versus the distance (z) from the median plane ($z = y_c$).

43. It is interesting to note that $A_\tau / \rho q_m$ has a maximum, halfway between wall and median plane, at a station where the curvature presents a minimum (Figures 4 and 5).

44. With regard to a logarithmic profile of velocity, the measurements were consistent not only in vicinity of the wall but further away, toward the median region of the cross section. Nonetheless, the plot u/u^* versus log η showed, for still higher values of η, a distinct swerving away of the empirical values from the straight line (Figure 6). Reichardt succeeded (Reference (m_1)) to eliminate such discrepancies by showing that, instead of u/u^*, a modified expression $u-u_i/u^*$ is a function of log η. The additional velocity u_i for the inner region of flow (hence the subscript i) was defined by the function (Figure 6):

$$\frac{u_i}{u^*} = \kappa^{-1} \ln \left\{ \frac{1.5\left(1 + y_c/r\right)}{1 + 2\left(y_c/r\right)^2} \right\} \qquad (15)$$

d. Complete Profile All Over the Cross Section

45. If the expression (15) for u_i/u^* is entered in the universal logarithmic law:

$$\frac{u - u_i}{u^*} = \kappa^{-1} \ln\left(1 + \kappa\eta\right) + c_1 \left\{1 - \exp\left(-\eta/\eta_1\right) - \eta/\eta_1 \exp\left(-b\eta\right)\right\} \qquad (16)$$

then the following unified law is holding in case of turbulent flow for the <u>velocity profile of the entire cross section:</u>

$$\frac{u}{u^*} = \kappa^{-1} \ln\left\{ \left(1 + \kappa\eta\right) \frac{1.5\left(1 + y_c/r\right)}{1 + 2\left(y_c/r\right)^2} \right\} + c_1 \left\{1 - \exp\left(-\eta/\eta_1\right) - \eta/\eta_1 \exp\left(-b\eta\right)\right\} \qquad (17)$$

The complete velocity profile of turbulent flow in a cylindrical tube is shown in Figure 7 for different Reynolds numbers.

46. It is easy to verify that the general relation (17) contains as particular cases both the velocity profiles that were previously derived for the wall region and the median region respectively. The former (equation (12)) occurs for $y_c/h \approx 1$. The latter, corresponding to large values of η, takes the form:

$$\frac{u}{u^*} = \kappa^{-1} \ln \eta + \frac{u_i}{u^*} = \kappa^{-1} \ln \left\{ \eta \frac{1.5\,(1 + y_c/h)}{1 + 2\,(y_c/h)^2} \right\} + c_1 \qquad (18)$$

47. The constant term c_1 is expected to depend on the shape of the cross section. It should be determined by experiment for the existing conditions of turbulent flow.

48. The unified velocity profile (17) of the prismatic duct is expected to hold as well for the turbulent flow along a flat plate. The latter agrees well with the former in proximity of the wall where the distribution of velocity depends mainly on the shear stress velocity:

$$u^* = (\tau_0/\rho)^{1/2}$$

49. In regions afar from the wall, however, individual features of the velocity profile may appear. The flow along a plate requires higher additional velocities (u_i) at the outer region than those used at the inner region of a duct for a truly logarithmic profile satisfying the complete set of boundary conditions.

Part II: The Profile of Mean Temperature

Introduction

50. This is the second part of a review on the characteristics of the turbulent flow with heat transfer in a prismatic duct of symmetric cross section or in the boundary layer of a flat plate.

51. For the laminar flow with such a geometry, the temperature profile has been rigorously computed some while ago. Yet, for the thermal field of turbulence no general analytic solution is known.

52. A main objective of this part is the recent progress in solving the turbulent case by integrating the temperature equation, numerically or graphically, in successive steps.

53. The turbulent flow is taken as fully developed, and as feeding upon the energy of a forced mean motion that is assumed stationary.

54. The solid boundary is postulated as smooth and of nearly uniform temperature.

55. It is found by integration how the fluid's temperature is dealt out normal to the wall in terms of either the distance from it or the velocity. The latter argument seems preferable for the higher Prandtl-numbers.

56. The variables are made dimensionless on a number of different bases. The divisor is either one of the border values (at wall or free stream) of the relevant quantity or its mean value over the boundary layer.

57. The molecular and turbulent part of both shear stress and heat flux density* are expressed in a function of the gradients of velocity and temperature respectively. Such combined expressions may lead to virtual solutions for the temperature in the integral form.

58. The integrands depend upon the fluid's influential properties: ρ and $P_r = c_p\mu/k$ assumed for the time being as nearly constant - and on three functional arguments. As such, the following dimensionless expressions are conveniently chosen (References (m) and (q)):

---

* flux = time rate of flow; the flux density is taken per unit of cross area.

a. The ratio: $\rho = A_\tau/\mu$ between the turbulent part of the shear stress (flux density of momentum) and its molecular counterpart. The exact form of " ρ " has been discovered recently by the inductive method (References (m_β), (m_γ), (m_δ)).

b. The dimensionless heat flux density: q/q_o, based on its value at the wall. The function q/q_o may be calculated in terms of the dimensionless wall distance (Reference (m_α)) by integrating the equation for the continuity of the heat flow, provided the shear distribution is known.

c. A function (m) to account for the experimental fact - inferred from References (g) and (i) - that turbulence is diffusing heat faster than momentum. This is expressed by the ratio $m = A_q / A_\tau$ between the coefficients: " A_q " for the thermodynamic effect of turbulence and " A_τ " for its mechanical action as apparent viscosity.

59. The form of this important function is unknown, though its boundary values are pretty sure ($1 \leq m < 2$), and average values over the boundary layers have been estimated (Figure 1) for particular cases (prismatic duct, plate).

60. Under the above premises, the integration for the profile of mean temperature has been recently performed in successive steps*. There is a rapid convergence, especially for not too small Pr-numbers.

61. By numerical or graphical integration, a satisfactory approximation is reached in one or two steps for air and fluids with higher Pr-numbers, while three or, at most, four steps are needed in the case of very small Pr-numbers (References (m_β), (m_δ), (q_β)).

62. The temperature field is discussed in view of the recently extended calculations of the heat transfer over a range of:

a. Very high values of the Pr-number (References (f) and (m), up to $Pr = 10^2$).

b. Very low Pr-numbers (References (m), (q) and (s) down to $Pr = 10^{-1}, 10^{-2}, 10^{-3}, 0$).

63. The former category corresponds to non-metallic liquids like water and oils. The latter covers conditions of metals in liquid state (mercury, molten metals and alloys). The

* Papers with analytical solutions for the single particular case: Pr = 1, m = 1, and independency of the fluid's properties from temperature, are not considered.

lower range became particularly interesting of late, in view of the necessity of removing heat in large amounts from nuclear furnaces, at a high temperature level, by applying molten metals as coolants.

64. The shape of the temperature profile appears to be highly dependent upon the generalized Pr-number: $Pr' = m\,Pr = (A_q/A_\tau)\,c_p\mu/k$ (Figures 9, 10 and 11), whereas it seems less affected by the Reynolds-number. However, the dimensionless heat flux density is found more dependent upon the Re-number.

65. Almost irrespective of both parameters (Pr' and Re), the average mean temperature for turbulent flow in a circular tube is found at a distance equal to 0.3 of the tube radius from the wall (References (p_α), (p_β), Jakob, M., and (q_β), Elser, K.)

66. By a sustained difference of temperature, between a solid wall and a fluid, a heat flow is caused between them in accordance with the formation of a thermal boundary layer. The steady state of such a field of enthalpy and temperature is investigated for generally specified geometric conditions at the boundaries.

67. In addition to the external cause of heat flow, there is the viscosity from molecular and turbulent origin to account directly or indirectly for internally released heat from dissipation of mechanical energy, when the fluid's velocity is sufficiently raised.

68. The thermal field is governed by the fluid's Pr-number, in addition to at least the Re-number of the flow, as long as the compression is not so high as to involve the Mach-number as well. Anyhow, the compressibility effects are presumed to be almost negligible. Moreover, neither combustion nor changes of phase are postulated. The latter restrictions allow for a similarity between the fields of enthalpy and temperature.

69. Since the early work of Osborne Reynolds (1874, 1890), the effects of turbulent agitation on diffusion of momentum and heat are currently expressed by analogous (Figure 1) formulas (G. I. Taylor, Prandtl, von Kármán) to those of corresponding molecular effects (Newton, Fourier). The flux density of both momentum (shear stress τ) and heat (q) are split into a molecular part and a turbulent counterpart.

$$\tau = \tau_m + \tau_t$$

$$q = q_m + q_t$$

70. The respective contributions are expressed* in function of coefficients and of gradients of velocity and temperature as follows:

$$\tau_t = A_\tau \frac{d\bar{u}}{dy} = -\rho \overline{u_f v_f}$$
$$q_t = -A_q \frac{d\bar{i}}{dt} = -c_p A_q \frac{dT}{dy} = c_p \rho \overline{T_f v_f} \tag{19}$$

71. At the wall (subscript 0) the turbulent agitation is completely impeded. This means $A_{q_0} = 0 = A_{\tau_0}$. Hence:

$$q_0 = -\left(k_0/c_{p_0}\right)\left(\frac{di}{dy}\right)_0 = -k_0\left(\frac{dT}{dy}\right)_0 \tag{19a}$$
$$\tau_0 = \mu_0 \frac{d\bar{u}}{dy}$$

72. Above expressions (19) are conveniently combined in a double ratio:

$$Pr' \simeq \frac{q_t/q_m}{\tau_t/\tau_m} = \frac{A_q}{A_\tau}\frac{c_p \mu}{k} = m\,Pr = -\frac{\overline{T_f v_f}}{\overline{u_f v_f}}\frac{du}{dT} Pr \tag{20}$$

73. By the dimensionless quantity "Pr'" or generalized Pr-number, the fluid's thermal property "Pr" is linked with a characteristic ratio "m" between two effects of turbulence. Therefore "Pr'" is a governing parameter par excellence for the turbulent flow when transport of both energy (heat) and momentum occurs.

74. Through turbulence, heat seems to spread faster than momentum. It is a fact concluded from extensive experimental evidence (References (d), (g), (m), (s)).

* Henceforth unbarred symbols, i, T, u, v, etc. are used whenever they, unmistakenly, refer to mean temporal values, unless it is otherwise expressly stated. However, by necessity, the vincula are maintained for denoting an average of a product, say $\overline{u_f v_f}$ or $\overline{T_f v_f}$

75. A measure of this phenomenon is the ratio "m" between the coefficients "A_q" for the thermodynamic effect of turbulence, and "A_τ" for its mechanical action by apparent viscosity.*

76. The ratio of these coefficients - $m = A_q / A_\tau$ is a characteristic function of the share taken by the turbulent effects with regard to the heat flux density and to the internal release of heat by viscosity. As such "m" seems to be dependent on the wall distance, and Pr' varies also with the local Prandtl-number. The latter changes with temperature - except for ideal gases - and the dependency is particularly strong for viscous fluids.

77. By neglect of the thermal variation of "Pr", the study would be limited to small changes of temperature for those fluids whose properties strongly depend upon it. Yet, by using an averaged Pr (over the boundary layer), some improvement would seem attainable.

78. No generally accepted theory about the mechanism of turbulence seems yet available for yielding the form of the function "m" over the viscous and thermal boundary films. Yet, it is pretty sure that: $1 \leq m < 2$; viz. "m" has the border value (1) at the wall (0): $m_0 = 1$, while at the free stream end of the boundary layer it seems to approach, - without quite reaching - the value: $m = 2$ of the free turbulence (hot jet or wake of heated rod).

79. As the shape of the function "m" is unknown, the estimated average values (Figure 1) over the boundary films are rather uncertain.

80. The measurements of temperature and velocity profiles for air, flowing over heated bodies, seem to show a dependency between the average value of "m" over the boundary layer and the layer's geometry. It matters whether the flow occurs in a duct (channel, tube) or around a submerged body (flat plate, missile, wing).

81. It may be of interest to examine whether - and to what extent - flow conditions such as the wall roughness and the pressure gradient (positive, negative or null) influence the distribution of "m".

82. In order to find the heat transfer at the wall, the velocity profile is needed in general. Similarity may exist

---

* Both A_q and A_τ , called also "Austausch" coefficients, have the dimensions of the dynamic coefficient (μ) of viscosity.

between the profiles of velocity, shear stress, and temperature, regardless of the Re-number, insofar as the finite boundary layer is laminar or the flow occurs at a Mach number high enough to reduce the width of the boundary layer to the order of a molecular mean free path.

83. On the contrary, for turbulent flow in subsonic or low supersonic range, the thickness of the viscous boundary layer is strongly dependent on the Re-number.

84. No yielding of the temperature profile may ensue by necessity from knowledge about the velocity profile. In general a separate integration should be indispensable for obtaining the former near the wall. This point would appear to vindicate devoting to the wall region of the thermal field a main part of our survey.

85. Moreover, it is important to know the temperature profile, because the heat transfer at the wall depends substantially on its shape. The heat transfer coefficient at the wall is:

$$h_{\theta} = \frac{k_0}{\Theta}\left(\frac{dT}{dy}\right)_0 = \frac{k_0}{U}\left(\frac{d\vartheta}{d\eta}\right)_0\left(\frac{d\bar{u}}{dy}\right)_0 = k_0\left(\frac{d\vartheta}{du}\right)_0\left(\frac{d\bar{u}}{dy}\right)_0 \tag{21}$$

86. For smaller Pr-numbers the temperature function $\vartheta(y/\kappa)$ seems more convenient, while for higher Pr-numbers, the function $\vartheta_u(u)$ appears preferable. It is interesting to note that, over a limited range of the Péclet number (Re Pr > 2500), the heat transfer might as well be derived from a general relation not linked to knowing expressly how the temperature is dealt out (Reference (m_1)). The latter possibility will be discussed in the third part of the review dealing specifically with the heat transfer at the wall.

87. Insofar as the influential properties of the fluid (c_p, μ, k and ρ) are nearly constant,* the energy equation will remain linear. Therefore a superposition should be permissible in regard to the thermal fields of external and internal origin. The latter is based on adiabatic walls (perfect thermal insulation) at equilibrium temperature.

* A dependency upon temperature - although weak - would seem more justified for the specific heat than for the heat conductivity.

88. With turbulence included in the aero-thermodynamic problem, there is an involved degree of intricacy in the combined flux of momentum and heat.

89. Three kinds of motion, basically different in nature, are interacting, i.e., the forced mean motion and two sorts of agitation on different levels. One of them operates on the tiny molecular scale by transport of momentum (viscosity) and heat (conductivity), while the turbulence occurs on a molar scale. Its relevant effects are: apparent viscosity and heat convection.

90. Under such involved conditions, no analytical integration is in sight for the partial differential equations that govern the conservation of momentum, mass and energy in the general turbulent flow with heat transfer. Neither are general solutions attainable yet for the simpler case of a plane mean motion (the turbulent one is always three-dimensional), as that occurring afar from the ends of prismatic ducts or along a flat plate.

91. However, with simplifications that were moderate and judicious, the thermal field was integrated numerically and graphically for the turbulent flow in a straight circular tube (References (m) and (q)).

92. In regard to a flat plate, a similar procedure seems promising only at a very large Re-number, for it allows the reduction of the plane temperature field to a mere one-dimensional problem. Other attempts to convert the two-dimensional, partial differential equation into an integrable, ordinary differential equation were shown to be not rigorous enough (Reference (q_α)).

Underlying Assumptions

93. To recapitulate, the present study is postulating:

a. A fully developed turbulent motion feeding upon the energy of a stationary, forced mean flow in a prismatic duct or along a flat plate.

b. A smooth solid wall throughout.

c. A heat flow caused by a difference of temperature between the solid wall and the fluid in the free stream in accordance with the shape of the thermal boundary layer.

d. A practically constant wall temperature over the entire surface of the solid boundary. Accordingly, the trans-

port of both heat and momentum in the respective boundary films is normal to the wall.

e. Flow conditions such as to not affect appreciably the density and other influential properties of the fluid (c_p, μ, k).

f. Negligible effects of radiation on the heat economy.

94. By postulate d, the resultant heat flow becomes one-dimensional in a convenient direction for making tractable the differential equations.

95. It would seem advisable to promote clarity and generality by not trying to write out too soon, under the integral, the imperfectly known functional expressions for $\rho = A_\tau/\mu$ and $m = A_q/A_\tau$. Besides, as to the function "m", as well as to the influential properties of the fluid (Pr, ρ), only average values may be anticipated adequately, for the time being. Later, the variable functional arguments of the mixed flow and of the fluid's properties might be taken into account, provided they are reducible to a judicious form and an acceptable one for keeping the integrals tractable without oversimplifying. It seems important to try to reach nontrivial solutions by non-too-arduous methods.

96. The above set of postulates proved apt to render soluble the differential equations for the turbulent flow with heat transfer by numerical or graphical methods of successive approximation (References (m_ρ), (m_s), (q_e)). The convergence proved rapid enough to yield satisfactory answers.

Reichardt's Refinements about the Turbulent Boundary Layer

97. A main incentive to the recent developments in the boundary layer theory was the trend toward approximating closer the temperature profile for turbulent flow and extending the solutions over a much wider range of Pr-numbers. In this connection much attention has been given of late to:

a. The boundary conditions for both the mean motion and the superposed turbulent one.

b. The observed continuity of decay of turbulence from the free stream to the immediate vicinity of the wall.

98. The early partition of the boundary layer, in one purely laminar part, adjacent to the wall, and one purely turbulent

outer part, allowed only a single value of the Pr-number: viz. Pr = 1 with:

$$A_q = A_\tau = \begin{cases} 0, & \text{in the laminar sublayer} \\ 1, & \text{in the turbulent layer} \end{cases}$$

99. Besides this limitation, such a dichotomy of the boundary layer implies a severe discrepancy in the form of a nonobservable singularity of the temperature profile at the outer limit of the purely laminar sublayer.

100. Later, the attempt to deal with heat transfer, by means of fluids with Pr $\neq$ 1, led to the well-known and still rather popular trichotomy of the boundary layer. This meant inserting a buffer layer (simple or composite) of a mixed - laminar and turbulent - nature, between the laminar sublayer and the turbulent outer region. However $A_q = A_\tau = 1$ was maintained in the two outer subdivisions, and the tripartite boundary layer did not escape from all the previous shortcomings by failing to satisfy certain boundary conditions.

101. To get rid of such contradictions, is the aim of the newest phase in the development of the boundary layer theory. It means abolishing the artifice of a purely laminar sublayer and inferring a refined distinction between a characteristic width (η_2) for the thermal profile and a characteristic one (η_1) for the viscous boundary layer. Except for Pr = 1, the boundary layer shows a different characteristic width in the fields of temperature and velocity.

102. The width of a film, in which a molecular effect is prevalent over its turbulent counterpart, is expressed as a limit value of a dimensionless distance from the wall, (say y/k or preferably: $\eta = y u^*/\nu$). Within a so defined edge of the velocity field, Newtonian viscosity prevails over the apparent one due to turbulence.

Similarly, for the temperature field, a characteristic thermal layer at the wall marks a region of prevalence for the Fourier conduction of heat by molecular agitation against the thermal convection from turbulence.

103. With

$$\delta = \frac{q_t}{q_m} \qquad \beta = \frac{\tau_t}{\tau_m} \qquad m = \frac{A_q}{A_\tau}$$

the generalized Pr-number is: $Pr' = m\, Pr\, . \, \delta/\beta$. Among the four functions: m, Pr, δ and β, only the last mentioned (ε/ν) appears as sufficiently explored over the boundary region (References (m) and (q)). However, with rough estimates for "m" and "Pr", the function "δ" may be approximated from:

$$\delta = (m)_{av} (Pr)_{av} \, \beta(\eta)$$

104. The exact form of "β" as a continuous function of "η" as discovered recently (References (m_β), (m_γ), (m_δ)) by the inductive method is based on the best available measurements and real boundary conditions. It also satisfies more completely the continuity requirements for both the mean motion and the turbulence.

105. With the dimensionless argument either in the form of a Re-number ($\eta = y u_*/\nu$) or $\sigma = \eta/\eta_m$ for the wall region or in the form y/a for the core (tube or duct) and the constants: $K = 0.4$ and $\eta_m = 7.15$ the following laws appear to satisfy the foregoing requirements:

106. <u>A-Region with</u> $\eta < 6$

Three terms are contributing to the function β as follows:

$$\beta(\eta) = \kappa\eta_m \sum_{i=0}^{3} c_i\, \sigma \tanh^i \sigma = \kappa\eta_m \left[\eta/\eta_m - \tanh \eta/\eta_m - \tfrac{1}{3} \tanh^3 \eta/\eta_m \right] \quad (22)$$

Variables: β, $\sigma = \eta/\eta_m$

i	0	1	3	≥ 5
c_i	1	$-1/5$	$-1/25$	0

Constants: $\kappa = 0.4$, $\eta_m = 7.15$

However, for extremely small values of η, the power series would seem to start with a term proportional to η^5. This means: $c_5 \neq 0$, $c_{i<5} = 0$. Then the above law would approach, according to Reference (m), for very small η, the form:

$$\beta \approx (3)\, 10^{-5}\, \eta^{5} \quad (22a)$$

107. B-Region with $6 < \eta < 30$.

Two terms only would contribute to the function $\rho(\eta)$, as follows:

$$\rho(\eta) = \kappa \eta_x \sum_{i=0,1} (c_i \, \sigma \tanh^i \sigma) = \kappa \left[\eta - \eta_x \tanh \eta/\eta_x \right] \qquad (23)$$

Variables: ρ, $\sigma = \eta/\eta_x$

i	0	1	≥ 2
c_i	1	$-1/\sigma$	0

Constants: $\kappa = 0.4$ $\qquad \eta_x = 11$

108. C-Region with $\eta > 30$.

This is the core of a tube or the median region of a channel. Here, "ρ" depends not only upon the dimensionless distance (y_c/r) from the tube-axis or the channel's medium plane, but also on the Re-number, indirectly, through the quantity:

$$\eta_r = \frac{r u^*}{\nu} = \frac{u^*}{2 U_{av}} Re = \frac{Re}{2 U_{av}/u^*}$$

109. The form of the function "ρ" (References (m_p), (m_v), (m_j), (t)) is here.

$$\rho = \frac{\kappa \eta_r}{3} \left[\frac{1}{2} + \left(\frac{y_c}{r} \right)^2 \right] \left[1 - \left(y_c/r \right)^2 \right] \qquad (24)$$

110. With regard to the effects of viscosity, a characteristic layer-width, "η_1" measured from the wall, may be defined for the prevalence in it of the molecular over the turbulent effect ($\tau_m > \tau_t$). Thus "η_1" is that value of the argument for which: $\rho(\eta) = 1$ (25)

111. Similarly, another characteristic width "η_2" may be referred to the heat flux density in the boundary layer. It marks a limit distance from the wall for prevalence in it of the molecular conduction over the turbulent convection of heat

($q_m \geq q_t$). Thus " η_2 " is, for $\delta = 1$, that value of η for which:

$$\beta(\eta_2) = \frac{1}{(m\,P_r)_{av}} = \frac{1}{P_{r\,av}'} \tag{26}$$

112. If Re is high enough, then " η_2 " ends within the outer part of the B-region (large η/η_m , hence tanh $\eta/\eta_m \approx 1$). Now η_2 is determined from the equation:

$$\beta(\eta_2) = \frac{1}{P_r'} = \kappa(\eta_2 - \eta_m) \tag{26a}$$

hence: $\eta_2 = \eta_m + \frac{1}{\kappa P_{r\,av}'} \approx \eta_1 + \frac{1}{\kappa P_{r\,av}'}$ (valid only for high Re)

113. It follows that, in general, by virtue of (25) and (26):

$$P_{r\,av}' < 1 \longrightarrow \eta_1 < \eta_2$$
$$P_{r\,av}' = 1 \longrightarrow \eta_1 = \eta_2$$
$$P_{r\,av}' > 1 \longrightarrow \eta_1 > \eta_2$$

114. With $\eta_m = 7.15$ for the extrapolated A-region, the value $\eta = \eta_1 = 10.7$ satisfies the limit width: $\beta(\eta_1) = 1$. Very nearly the same characteristic width for the viscous layer --- viz. $\eta_1 = 10.8$ --- is attained by using the β-law of the B-region with $\eta_m = 11$.

115. For a given shear stress distribution, η_1 is nearly constant. On the contrary the characteristic width (η_2) of the thermal layer is variable as dependent on $Pr' = m\,P_r$.

116. It is interesting to note that for $Pr' \gg 1$, the corresponding low values of the function β are obtained with accordingly small values of $\eta_2 \ll \eta_1$. That means an extremely thin characteristic width for the heat conducting layer with a prevalence of turbulent heat convection within the layer ($\eta \leq \eta_1$) of prevailing molecular viscosity.

117. Example: for $Pr' = 500$ and $\beta = 0.01$: $\delta = \beta\ Pr' = 5$, i.e. $q_t = 5\,q_m$, although $\tau_t = (0.01)\,\tau_w$.

118. This illustrates clearly the fallacy of conceiving a purely laminar sublayer of finite thickness.

119. On the other hand, for $Pr' \ll 1$, the corresponding high values of the function β lead to $\eta_2 \gg \eta_1$. Thus the heat transfer by metallic fluids operates mainly by molecular heat conduction even at values of η at which the flux density of momentum is mainly a turbulent phenomenon.

120. Example: for $Pr' = 0.01$ and $\beta = 1$: $\delta = \beta Pr' = 0.01$ i.e. $q_m = (100) q_t$ at the limit ($A_\tau = \mu$) of the viscous layer. This would hold for a metallic liquid like mercury as heat transferring medium.

121. The early assumption of a common mechanism of turbulence to account for both thermal (q_t) and mechanical (τ_t) effects implied as well $(m)_{av.} = 1$. By such a theory the dichotomy of the boundary layer in a purely laminar and a purely turbulent outer part, appeared satisfactory for a fluid with Pr = 1. That would be steam or a polyatomic gas, at least quadratomic.

122. However, according to the present views: $m_{av} \approx 1.4$, for the flow near the wall of a prismatic duct. It follows that the restricted validity of the old theory deserves being interpreted in the light of the latest conceptions. These show the old theory holding not for a fluid with Pr = 1, but for one with Pr' = 1, implying: $Pr = Pr'/m_{av} = 1/1.4 = 0.72$. This fluid is, of course, air and not steam.

123. The solid background of the new theory is the continuity in the decay of the turbulent effects from the free stream to the wall, with, in general, $\eta_1 \neq \eta_2$. The new approach has already allowed for an extension of knowledge about the temperature distribution and heat transfer over a very wide range of the Pr-number: $10^{-3} \leq Pr' \leq 10^3$

124. In the following table the salient features of the boundary layer theory are juxtaposed with regard to three milestones of its evolution.

TABLE

Three Milestones of Progress in Boundary Layer Theory

Period	Main Initiators	Characteristics	Heat Transfer Problem soluble for
Early: 1904/ 1938	Prandtl, Taylor, etc.	Dichotomy of the boundary layer in a purely laminar region near the wall and a fully turbulent, outer region. Prerequisites of this theory are: (1) Similarity of boundary conditions in the equations of momentum and heat. (2) No pressure gradient (flow along flat plate). - Otherwise a fictitious heat source arrangement should be superposed. -	Fluid with Pr' = 1. However, originally, both m_{av} and Pr were taken as unity. With the actual value: $m_{av} > 1$ the original results hold for the single value: $Pr = \frac{1}{m_{av}} < 1$ $= 0.75$ for $m = 1.4$
Middle: 1938/ 1949	Von Kármán, Reichardt etc.	Trichotomy of the boundary layer in one purely laminar wall region, one intermediate buffer layer (simple or composite) and one outer, purely turbulent region.	Fluids with: Pr' > 1 and $m_{av} = 1$ The theory holds on a rather limited range of values for the Pr-number.
Actual: since 1949	Reichardt, Elser, etc.	Continual decay of the effects of turbulence from the free stream to their suppression at the wall. A characteristic width (η_1), is referred to the transport of momentum. This width is defined as that distance from the wall within which Newtonian shear stress prevails over the apparent one from turbulence. Similarly, for the thermal boundary layer, another characteristic width (η_2) marks the region of prevalence for the heat transport by molecular conduction over the counterpart due to turbulent convection. Except for Pr' = 1, there is: $\eta_1 \neq \eta_2$	Fluids with Pr'>>>1 and such with Pr'<<<1, and variable m on the wide range of: $10^{-3} \leq Pr' \leq 10^{3}$

Differential Equation for the Temperature Distribution

125. The differential equation for the thermal profile near the solid boundary may assume either form (A) of the temperature gradient: $\frac{dT(y)}{dy} = F(y)$

or form (B) as: $\frac{dT(u)}{du} = \phi(u)$

126. In the form (A) the distance from the wall is chosen as argument of the sought function with either both variables being dimensional in T (y) or both being reduced to dimensionless varieties like $\vartheta(y/r)$ or $\vartheta(yu^*/\nu)$. The form (A) seems preferable for fluids with Pr' << 1.

127. The form (B) of the temperature function has the velocity as argument. Again, either both variables are taken in the dimensional form T (u) or dimensionless varieties are used for both in $\vartheta(\varphi)$ or $\vartheta(\Psi)$. In most cases of practical interest: $\Psi/\varphi = U/u^* \approx 20$ to 35 (Reference (m_β)).

128. The form (B) seems preferable, for Pr' >> 1, because of the rapid decay of the integrand function with the dimensionless distance (y/r) from the wall.

a. Form A

129. With reference to formulas (19), the total flux density for the heat may be expressed as follows:

$$q = (1 + \delta)\, q_m$$

which, with: $\delta = \beta\, Pr'$ and $q_m = k\frac{dT}{dy}$ yields:

$$q = (1 + \beta Pr')\, k\, \frac{dT}{dy}$$

130. By accounting for the boundary value at the wall: $q_o = k_o\left(\frac{dT}{dy}\right)_o$ and marking a symbolic integration, one obtains:

$$T - T_o = \left(\frac{dT}{dy}\right)_o \int_0^{y/r} \frac{k_o}{k}\, \frac{q/q_o}{(1+\beta Pr')}\, d(y/r) \qquad (27)$$

131. With dimensionless variables: $\vartheta = (T - T_o)\theta^{-1}$ and y/r, the relation is transformed into:

$$\vartheta = \left(\frac{d\vartheta}{d\eta}\right)_0 \int_0^{y/\tau} \frac{k_0}{k} \frac{q/q_0}{(1+\beta P_z')} \, d(y/\tau) \tag{28}$$

$$0 \leq y/\tau \leq 1 \longrightarrow 0 \leq \vartheta \leq 1$$

b. Form B

132. With reference to formulas (19) the expressions for q_m and τ_m may be combined into:

$$\frac{dT}{du} = \frac{\mu}{k} \frac{q_m}{\tau_m} \tag{29}$$

On the other hand:

$$\frac{q - q_0}{q_m} = \beta P_z'$$

And solving for q_m:

$$q_m = \frac{q}{1 + \beta P_z'}$$

133. By substituting this expression for q_m, the equation (29) is transformed into:

$$\frac{dT}{du} = \frac{\mu}{k} \frac{q}{\tau_m + \tau_t P_z'} = \frac{\frac{\mu}{k} q/\tau}{1 + \frac{\beta}{1+\beta}(P_z' - 1)} \tag{30}$$

134. By taking in account the boundary conditions at the wall ($\tau_{t_0} = 0$, $\tau_0 = \tau_{m_0}$):

$$\frac{dT}{du} = \frac{\mu_0}{k_0} \frac{q_0}{\tau_0}$$

The symbolic integration of equation (30) yields:

$$T - T_0 = \left(\frac{dT}{du}\right)_0 \int_0^u \frac{\gamma \, \xi \, du}{1 + \frac{\beta}{1+\beta}(P_z' - 1)} = \left(\frac{dT}{du}\right)_0 \int_0^u \frac{\gamma \, \xi (1+\beta) \, du}{1 + \beta P_z'} \tag{31}$$

for

$$\gamma = \frac{\mu}{\mu_0} \frac{k_0}{k} = \frac{P_z}{P_{z_0}} \frac{c_{p_0}}{c_p} \quad \text{and} \quad \xi = \frac{q}{q_0} \frac{\tau_0}{\tau}$$

135. With the influential properties of the fluid assumed constant, i.e. with $\xi \approx 1$, and a shift to dimensionless variables ($\vartheta = |T - T_w| \Theta^{-1}$ and $\varphi = u/V$), the differential equation (31) is reduced to:

$$\vartheta = \left(\frac{d\vartheta}{d\varphi}\right)_0 \int_0^{\varphi} \frac{\xi(1+\beta)}{1+\beta Pr'}\, d\varphi \tag{32}$$

Integration of the Temperature Equation

136. The integration of the temperature profile is discussed for the form (B) of the differential equation in its dimensionless form $\vartheta(\varphi)$. The same method, however, is easily applicable to form (A) as well.

137. In a slightly modified way, equation (32) may read:

$$Pr'\vartheta = \left(\frac{d\vartheta}{d\varphi}\right)_0 \left[\int_0^{\varphi} \xi\, d\varphi + \int_0^{\varphi} \frac{(Pr'-1)\,\xi}{1+\beta Pr'}\, d\varphi\right] \tag{32a}$$

and by means of the identity:

$$\frac{Pr'(1+\beta)}{1+\beta Pr'} = 1 + \frac{Pr'-1}{1+\beta Pr'}$$

equation (32a) is transformed into:

$$Pr'\vartheta = \left(\frac{d\vartheta}{d\varphi}\right)_0 \left[\int_0^{\varphi} \xi\, d\varphi + \int_0^{\varphi} \frac{(Pr'-1)\,\xi}{1+\beta Pr'}\, d\varphi\right] \tag{33}$$

138. The important function $\xi = \dfrac{q/q_0}{\tau/\tau_0}$ may be expressed by a Binomial:

$$\xi = 1 + \lambda \tag{34}$$

with "λ" as a dimensionless quantity which nearly vanishes at the wall ($\lambda_o \approx 0$), while assuming a value $0 < \lambda < 1$ afar from the wall.

139. The value $\xi = 1 + \lambda_e \approx 1$ may be used as a first approximation to yield (q/q_o), which, close to the wall, is identical to the function τ/τ_o . The latter is a linear function of the distance from the wall for the prismatic ducts that are considered:

$$\tau/\tau_o = 1 - y/r = y_c/r \tag{35}$$

140. If τ/τ_o is known, the function q/q_o may be found from the continuity of heat flow by balancing out the amounts of heat involved in flow-directions parallel to the mean motion and across. This reflects a similarity between the temperature profiles at different stations along a cylindrical duct. For instance, this means for a circular tube, that, over the radius, the flux density of heat is accounting for a proportionality between amounts of heat that are transferred radially and axially.

141. Thus, the dimensionless heat flux density (q/q_o) may be expressed by the following integrals (Reference (m_α)):

Tube:
$$q/q_o\,(1 - y/r) = 1 - \frac{\int_0^{y/r} \vartheta \varphi\,(1 - y/r)\, d(y/r)}{\int_0^1 \vartheta \varphi\,(1 - y/r)\, d(y/r)} \tag{36a}$$

Channel:
(rectangular)
$$q/q_o = 1 - \frac{\int_0^{y/r} \vartheta \varphi \, d(y/r)}{\int_0^1 \vartheta \varphi \, d(y/r)} \tag{36b}$$

142. In replacement of $\dot{q}$ by $\frac{q/q_0}{\tau/\tau_0}$, the equation for the temperature profile reads:

$$\vartheta = \left(\frac{d\vartheta}{d\varphi}\right)_0 \int_0^{\varphi} \frac{q/q_0\,(1+\beta)}{\tau/\tau_0\,(1+\beta\,Pr')}\,d\varphi \qquad (37)$$

Provided the velocity profile (φ) and shear distribution (τ/τ_0) are known functions, the temperature profile (ϑ) may be determined from the equations (36) and (37) by a method of successive approximations.

143. As first approximation (ϑ_1) for the thermal field in the boundary layer, a functional form may be chosen to correspond to $\lambda = \lambda_0 = 0$ i.e. $(q/q_0)_1 = \tau/\tau_0$

Thus: $$\vartheta_1 = \left(\frac{d\vartheta}{d\varphi}\right)_0 \int_0^{\varphi} \frac{1+\beta}{1+\beta\,Pr'}\,d\varphi \qquad (38)$$

" β " being a known function of y/h and φ, while, for the unknown function Pr' = mPr, a convenient average value over the boundary layer may be estimated.

144. If ϑ_1 is entered in one of the equations (36a) or (36b) together with the known velocity distribution $\varphi\,(y/h)$, a second approximation $(q/q_0)_2$ can be determined regarding the heat flux density, say for the channel:

$$\left(q/q_0\right)_2 = 1 - \frac{\int_0^{y/h} \vartheta_1\,\varphi\; d(y/h)}{\int_0^1 \vartheta_1\,\varphi\; d(y/h)}$$

A second approximation (ϑ_2) for the thermal field will be:

$$\vartheta_2 = \left(\frac{d\vartheta}{d\varphi}\right)_0 \int_0^{\varphi} \frac{(q/q_0)_2\,(1+\beta)}{\tau/\tau_0\,(1+\beta\,Pr')}\,d\varphi \qquad (38a)$$

$$I_2 = \int_0^{\varphi} \frac{\lambda P_z'(1+\beta)}{1+\beta P_z'}\, d\varphi \qquad (40b)$$

a. Integral I_1

149. In the first part of this review, the dimensionless velocity profile has been expressed as $\psi(\eta)$, with $\psi = u/u^*$.

150. By transforming the integral I_1 from φ to ψ, and using the identity:

$$\frac{d\psi}{d\eta} = \frac{\tau/\tau_0}{1+\beta} \quad \text{and} \quad d\varphi = \frac{du}{U} = \frac{u^*}{U}\, d\psi$$

we obtain:

$$\int_0^{\varphi} \frac{d\varphi}{1+\beta P_z'} = \frac{u^*}{U}\int_0^{\psi} \frac{d\psi}{1+\beta P_z'} = \frac{u^*}{U}\int_0^{\eta} \frac{(\tau/\tau_0)\, d\eta}{(1+\beta)(1+\beta P_z')} \qquad (41)$$

151. As the function $\beta = A\varepsilon/\mu$ is sharply growing with the dimensionless distance in the Reynolds form: $\eta = yu^*/\nu$, the integrand is declining with growing η. Hence $I_1 \neq 0$ only very near to the wall where $\tau \approx \tau_0$. This is expressed in closely approximating the integral I_1 with " a_η ".

$$I_1 \approx a_\eta = \int_0^{\eta} \frac{d\eta}{(1+\beta)(1+\beta P_z')} \qquad (42)$$

b. Integral I_2

$$I_2 = \int_0^{\varphi} \frac{\lambda Pr'(1+\beta)}{(1+\beta Pr')} d\varphi$$

152. Very near to the wall: $\lambda_0 \approx 0$ i.e. $I_{20} \approx 0$. Afar from the wall: $\beta \gg 1$; hence the unity becomes negligible as compared to β and to $\beta Pr'$ (for Pr' not too small), and for high values of y (or one of its dimensionless varieties) we obtain:

$$\frac{Pr'(1+\beta)}{1+\beta Pr'} \approx 1$$

and#

$$\varepsilon_\varphi = \int_0^{\varphi} \lambda \, d\varphi \approx I_2 \tag{43}$$

153. By considering the relevant boundary conditions it may be shown that the integral: $\varepsilon = \int \lambda \, d\varphi$, over the whole boundary layer would be negligibly small for a flow past a flat plate. The same integral would be less negligible for a two dimensional flow in an open duct, and still larger for the flow in a tube. This means:

$$\varepsilon_{\text{flat plate}} \ll \varepsilon_{\text{open channel}} < \varepsilon_{\text{tube}}$$

Integral Form of the Temperature Profile

154. With the expressions ε_λ and ε_φ, the integral form for the temperature profile (40) becomes:

Mostly the symbolism of Reference (m) is adopted. The exceptions include cases where a conflict had to be prevented with well accepted American practice, e.g. by using "k" instead of "λ" for the heat conductivity.

$$Pr'\,\vartheta = \left(\frac{d\vartheta}{d\varphi}\right)_0 \left[\varphi + \frac{u^*}{U}(Pr'-1)\,a_\eta + \varepsilon_\varphi\right] \qquad (44)$$

155. In denoting simply by "a" and "ε" (without subscript) the respective values of the two integrals at a limit distance from the wall for which (y = r): $\vartheta = \varphi = 1$, we obtain as gradient of the temperature profile at the wall:

$$\left(\frac{d\vartheta}{d\varphi}\right)_0 = \frac{Pr'}{1 + \frac{u^*}{U}(Pr'-1)\,a + \varepsilon} \qquad (45)$$

156. Finally the integral form of the temperature profile reads:

$$\vartheta = \frac{\varphi + \frac{u^*}{U}(Pr'-1)\,a_\eta + \varepsilon_\varphi}{1 + \frac{u^*}{U}(Pr'-1)\,a + \varepsilon} \qquad (46)$$

157. Similarly, with $\psi = u/u^*$:

$$\vartheta/\varphi = \frac{1 + (Pr'-1)\frac{a_\eta}{\psi} + \varepsilon_\varphi/\varphi}{1 + (Pr'-1)\frac{u^*}{U}a + \varepsilon} \qquad (46a)$$

Characteristic plots concerning the thermal boundary layer, as obtained by integration, are shown in the Figures 8 - 11 (References (m_o) and (m_f)). The following indications are to supplement their captions:

Figure 8 These profiles are a second approximation, $(q/q_0)_2$ of the heat flux density.

The dash-dotted parabola represents the heat flux density for the laminar flow, as obtained by a similar numerical integration in two steps.

Figure 9 The first approximation (dashed lines) refers to a flat plate, the second one (solid lines) to a channel duct.

The small rings mark the points whose abscissae measure the width " η_2 " of the prevalently heat conducting layer. The single abscissae " η_1 " - independent of Pr' - is a limit of prevalence for the molecular viscosity.

Part III: The Turbulent Heat Transfer at Smooth Walls

Introduction

158. With a geometry of flow as considered here (prismatic duct or plate), the analytic expressions for the function $\epsilon_r/\mu = f(\eta)$ over the several regions of the boundary layer - as recently discovered (References (m), (q) and (t)) - have allowed to determine more rationally the profile of mean values for both velocity and temperature. The distributions may be approximated analytically with a local, differential formulation by integrating hereafter either directly for the velocity (Part I of this survey) or stepwise for the temperature profile (Part II).

159. A chief objective at present is to discuss the heat transfer problem for turbulent flow as handled by means of an iterative integration.

160. The theoretical results from a local approach are compared with those of global, semi-empirical relations for the heat transfer coefficient at the wall. The latter appear as a product of powered similarity parameters as factors.

161. With moderate restrictions, as specified in References (m) and (t), two integrating varieties of the method are evolved:

a. The computing of heat transfer is based on determining first the temperature distribution over the boundary layer (References (m), (q), (s) and (t)) by successive steps of integration; then, the slope of the thermal profile at the wall is computed and entered into any one of several accepted forms of the heat transfer coefficient, e.g. the dimensional one (h) or the dimensionless varieties (h_{ij} Nu, St, etc.).

b. The heat transfer at the wall may be determined directly from a general relation for the temperature gradient at the wall, without full knowledge of an integrated temperature profile.

162. The first method is based on evaluating all of the integrals involved in the temperature profile. For a two step approximation, their number is four. Two of them are indefinite integrals, and two are definite.

163. The second method implies the evaluation of the two definite integrals only. These are to a close approximation:

$$a = \int_0^{\Psi} \frac{d\psi}{1 + \beta\, Pr'} \qquad \text{and} \qquad i = \int_0^1 \left(\frac{q/q_0}{\tau/\tau_0} - 1 \right) d\varphi$$

with $\psi = u/u^*$, $\Psi = U/u^*$, $\varphi = u/U$, $\Phi = 1$

164. For them, the limits of integration are at both ends of the boundary film. Integrals with a variable upper limit would be irrelevant here, although they are indispensable for tracing the complete temperature profile from point to point along the normal to the wall.

165. These methods of successive integration are of practical interest as long as a so-called universal law holds for the function $\beta(\eta)$. This would mean an independency of β from the individual flow conditions in regard to the geometry of wall configuration and properties of the fluid. A restriction in this sense is associated with the Péclet number not being allowed to understep a certain lower limit, viz.

$$P\acute{e} = Pr\, Re \geq 2500$$

166. This limitation corresponds to flow conditions which combine in such a way as to keep the characteristic width " η_L " of the prevalently heat conducting layer within the range of validity of those forms (A and B) of the function $\beta(\eta)$ that hold near the wall. Beyond this range, i.e. for $\eta_L > 30$ (References (m) and (t)), " β " would depend not only upon the wall distance (say: η in dimensionless form), but upon the Re-number as well, through the quantity: $\eta_L = Re/2\Psi_m$ that is involved in the C-form of the function β (References (m) and (t)).

167. A check on whether or not the actual Re is high enough for a practical validity of these methods of integration would seem especially desirable for fluids with a very small Pr-number, which yield high values of the ratio 2500/Pr.

168. On the other hand, for a very high Pr-number, the integrals involved in the heat transfer calculations show a trend of a considerable simplification. The numerical values of the heat transfer coefficient, as obtained by successive integrations, were found in good agreement with those based on semi-empirical power relations as reported in Reference (m) for the high Pr-numbers and in Reference (q) for the low Pr-numbers. Thus the compared results cover a wide range of the generalized Pr-number.

Analytic Approach by Iterative Integration

169. The profile of the dimensionless mean temperature - say $\vartheta(y/r)$ of a circular tube with radius r - may be determined for nearly constant average properties of a turbulent fluid, by successive steps of integrating the following two equations:

$$q/q_0\,(1 - y/r) = 1 - \frac{\int_0^{y/r} \vartheta\varphi\,(1 - y/r)\,d(y/r)}{\int_0^1 \vartheta\varphi\,(1 - y/r)\,d(y/r)} \tag{47}$$

$$\vartheta = \left(\frac{d\vartheta}{d\varphi}\right)_0 \int_0^{\varphi} \frac{(q/q_0)\,(1+\beta)}{(\tau/\tau_0)\,(1+\beta\,Pr')}\,d\varphi \tag{48}$$

170. Two unknown functions: q/q_0 and ϑ are thus interrelated to each other and to three nearly known expressions: $\beta = A_\tau/\mu$, $\varphi = u/u^*$ and τ/τ_0. A dimensionless either wall distance y/r or $\eta = (y/\nu)(\tau_0/\rho)^{1/2}$ or velocity (φ or Ψ) is a common argument for all of the five functions involved in the above integral equations. As to $Pr' = (A_q/A_\tau)\,c_p\mu/k$ it is a parameter for which a mean value over the boundary layer is used according to the geometry of the flow (m_{av}) and the nature of the fluid ($(Pr')_{av}$).

171. As first step of approximation, the expression $q/q_0 \approx \tau/\tau_0$ is substituted for q/q_0 in equation (48) which then may yield by a single integration, the first approximation, ϑ_1, for the temperature profile.

172. The second step consists in substituting ϑ_1 for ϑ into the two integrands of equation (47), thus obtaining $(q/q_0)_2$ after having evaluated the two integrals. The second approximation of the heat flux density is used in turn for integrating equation (48) once more. This yields the second approximation (ϑ_2) of the temperature distribution.

173. By substituting iteratively the successive expressions for (q/q_o) and ϑ_i from one into the other integral equation, the n^{th} step of approximation for the temperature profile may be reached. It would imply the evaluation of $1 + 3\,(n-1)$ integrals involved in the general expression:

$$\vartheta_n = \left(\frac{d\varphi}{d\vartheta}\right)_o \int_o^{\varphi} \frac{(q/q_o)_n\,(1+\beta)}{(\tau/\tau_o)\,(1+\beta P_r')}\,d\varphi \tag{49}$$

174. Except in cases of very low Pr-numbers (molten metals and alloys, etc), two steps of approximation, i.e. n = 2, should suffice. Then, the number of integrals to be evaluated would be 1 + 3 (2 — 1) = 4.

175. As to the choice of the dimensionless argument, either a relative velocity ($\varphi = u/U$; $\psi = u/u^*$) or a relative distance from the wall ($\eta = y u^*/\nu$; for the circular tube: y/r) have been used according to conditions of flow.

176. With

$$\Omega = q_m/q = \frac{1}{1+\beta P_r'} = \frac{1}{1 + (A_\tau/\mu)_m \dfrac{c_p \mu}{k}}$$

two out of four integrals, for n = 2, have a variable upper limit within the boundary layer, while the extent of integration, for the other two, reaches the free-stream value of the relevant argument.

177. The two indefinite integrals may show one of the following forms:

$$I_1 = \int_0^{\varphi} \Omega\,d\varphi = \varphi^* \int_0^{\psi} \Omega\,d\psi = \int_0^{y/r} \Omega\,\frac{d\varphi}{d(y/r)}\,d(y/r) = \varphi^* \int_0^{\eta} \frac{(\tau/\tau_o)\,\Omega\,d\eta}{1+\eta} \tag{50}$$

$$I_2 = \int_0^{\varphi} \lambda(1+\beta)\,\Omega\,P_r'\,d\varphi \approx \int_0^{\varphi} \lambda\,d\varphi = \int_0^{\varphi}\left(\frac{q/q_o}{\tau/\tau_o} - 1\right)d\varphi = \varphi^* \int_0^{\psi} \lambda\,d\psi \tag{51}$$

178. The corresponding definite integrals with subscript denoting an extension of the integration over the entire boundary layer are:

$$I_{1\ell} = (I_1)_{\text{free stream}} = \int_0^1 \Omega \, d\varphi = \varphi^* \int_0^{\Psi} \Omega \, d\psi = \int_0^1 \Omega \frac{d\varphi}{d(\tau/\tau_0)} d(\tau/\tau_0) \tag{52}$$

$$= \varphi^* \int_0^H \frac{(\tau/\tau_0)\Omega \, d\eta}{1+\beta}$$

$$I_{2\ell} = (I_2)_{\text{free stream}} = \int_0^1 \lambda \, d\varphi = \varphi^* \int_0^{\Psi} \lambda \, d\psi = \int_0^1 \lambda \frac{d\varphi}{d(\tau/\tau_0)} d(\tau/\tau_0) = \varphi^* \int_0^H \frac{\lambda \, d\eta}{1+\eta} \tag{53}$$

179. The expression for the heat flux density at the wall,

$$q_0 = k \left(\frac{dT}{dy}\right)_0 \tag{54}$$

may be transformed, by the use of dimensionless variables:

$$\vartheta = \frac{T-T_0}{\Theta}, \quad \varphi = u/U, \quad \psi = u/u^*, \quad \tau_0 = \rho u^{*2} \quad \text{into}$$

$$q_0 = \frac{k}{\mu} \frac{\Theta}{U} \tau_0 \left(\frac{d\vartheta}{d\varphi}\right)_0 \tag{55}$$

180. As shown in the second part of this review, the slope of the thermal profile at the wall may be given the form:

$$\left(\frac{d\vartheta}{d\varphi}\right)_0 = \frac{P_r'}{1 + (P_r'{}_{\infty} - 1) I_{1\ell} + I_{2\ell}} \tag{56}$$

181. Similarly it was shown that, in a two step approximation, the expression for the profile of the mean dimensionless temperature (ϑ) is finally:

$$\vartheta = \frac{(d\vartheta/d\varphi)_0}{P_r'} \left\{ \varphi + (P'_{r_{av}} - 1) I_1 + I_2 \right\} = \frac{\varphi + (P'_{r_{av}} - 1) I_1 + I_2}{1 + (P'_{r_{av}} - 1) I_{1\ell} + I_{2\ell}} \qquad (57)$$

182. By replacing in relation (55), the slope $(d\vartheta/d\varphi)_0$ by the expression (56), one obtains:

$$q_0 = \frac{\frac{k\theta}{\mu U} \tau_0 m_{av} P_r}{1 + (P'_{r_{av}} - 1) I_{1\ell} + I_{2\ell}} = \frac{\rho c_p U \theta \varphi^{*2} m_{av}}{1 + (P'_{r_{av}} - 1) I_{1\ell} + I_{2\ell}} \qquad (58)$$

183. The accepted forms for the heat transfer coefficient at the wall are either dimensional (h_i without a superscript) or dimensionless varieties as: h'_{ij} , $Nu_i = h_i d/k$

$$St = \frac{Nu}{R_e P_r} = \frac{Nu}{P\ell} \; ; \qquad St' = \frac{Nu}{R_e P_r'} = \frac{Nu}{P\ell'}$$

184. The subscript i refers to the temperature difference (ΔT_i) used as a divisor in deriving the dimensional heat transfer coefficient from the heat flux density at the wall (0). The second subscript (j) indicates the reference velocity. Below are listed some of the current varieties in forming the dimensional (h_i) and the dimensionless form (h'_{ij}) of the heat transfer coefficient at the wall:

Dimensional form:

$$h_i = \frac{q_0}{\Delta T_i} \begin{cases} h_{a} = q_0 / T_a - T_0 \\ h_{u} = q_0 / T_u - T_0 \\ h_{\vartheta_u} = q_0 / \vartheta_u \\ h_{\theta} = q_0 / \theta \end{cases} \qquad (59)$$

Dimensionless form: $h'_{ij} = \dfrac{h_i}{\rho c_p U} = \dfrac{q_o}{\rho c_p U \Delta T_i}$ (60)

185. Two varieties are used for the second subscript " j " by taking as reference velocity either the free-stream value (U) or the shear stress velocity ($u^* = (\tau_o/\rho)^{1/2}$), viz:

$$h'_{\theta U} = \frac{q_o}{\rho_o c_{p_o} \theta U} = \frac{\varphi^{*2}}{P_{r_o}} \left(\frac{d\vartheta}{d\varphi}\right)_o = \frac{\varphi^{*2}\, m_{av}}{1 + (P_{r_{av}} - 1) I'_{1\epsilon} + I_{2\epsilon}} \quad (61)$$

and similarly:

$$h'_{\theta u^*} = \frac{h'_{U\theta}}{\varphi^*} = \frac{q_o}{\rho_o c_{p_o} \theta u^*} = \frac{\varphi^*}{P_{r_o}} \left(\frac{d\vartheta}{d\varphi}\right)_o = \frac{\varphi^* m_{av}}{1 + (P_{r_{av}} - 1) I'_{1\epsilon} + I_{2\epsilon}} \quad (62)$$

with

$$I'_{1\epsilon} = \varphi^* \int_0^{\psi} \frac{P_{r}' - 1}{P_{r_{av}}' - 1} \frac{d\psi}{1 + \beta P_r'} = \int_0^1 \frac{P_r' - 1}{P_{r_{av}}' - 1} \frac{d\varphi}{1 + \beta P_r'} = \int_0^1 \frac{P_r' - 1}{P_{r_{av}}' - 1} \frac{1}{1 + \beta P_r'} \frac{d\varphi}{d(y/h)} d(y/h)$$

186. Except h'_{ij}, two more dimensionless varieties are in use for the heat transfer coefficient, viz. the Nusselt number (Nu) and the Stanton number (St). These may be expressed as follows:

$$Nu = \frac{h_u d}{k} = \frac{q_o d}{(T_u - T_o) k} = \frac{h' \theta u^* P_{r_o} Re}{\vartheta_u \varphi_m}$$

(63)

$$St = \frac{Nu}{Re\, P_r} = \frac{q_o}{\rho_o c_{p_o} u_m \Delta T_m} = \frac{h_u}{\rho_o c_{p_o} u_m} = \frac{h'_{u\theta}}{\varphi_m \vartheta_u} = \frac{h'_{U\theta} U \theta}{u_m T_u}$$

$$St' = \frac{Nu}{Re_m P_r}$$

187. A general formula for the Nu-number may be derived by combining the following two equations:

$$Nu\ \Delta T_u^* = \frac{2r q_o}{k_o}$$

and

$$\vartheta = \left(\frac{d\vartheta}{d(y/r)}\right)_o \int_o^{y/r} \frac{k_o}{k} \frac{q/q_o}{1+\beta P_r'} d(y/r)$$

(Reference (m_8)) from the boundary condition $\vartheta = 1$ for $y/r = 1$, the temperature gradient at the wall is:

$$\left(\frac{d\vartheta}{d\varphi}\right)_o = \frac{1}{\int_o^1 \frac{k_o}{k} \frac{q/q_o}{1+\beta P_r'} d(y/r)}$$

188. On the other hand, with

$$\Delta T_u = T_u \vartheta_u \quad \text{and} \quad q_o = k_o \left(\frac{dT}{dy}\right)_o$$

It follows that:

$$N_u\ \vartheta_u = 2\left(\frac{d\vartheta}{d(y/r)}\right)_o = \frac{2}{\int_o^1 \frac{k_o}{k} \frac{q/q_o}{1+\beta P_r'} d(y/r)} \qquad (64)$$

189. The main integral in the denominator of formula (62) may be approximated by:

$$\left(I_{16}\right)_{approx.} = \int_o^1 \frac{d\varphi}{1+\beta P_r'} = \int_o^1 \frac{1}{1+\beta P_r'} \frac{d\varphi}{d(y/r)} d(y/r)$$

190. Thus for a logarithmic profile of velocity, implying $\frac{d\varphi}{d(y/r)} y = \text{const} = g$ the integral becomes:

$$\left(I_{16}\right)_{approx.} \approx \int_o^1 \frac{g}{y(1+\beta P_r')} d(y/r)$$

191. It is interesting to note that the integrand of $(I_{i\ell})_{approx.}$ presents a more rapid decay with the dimensionless wall distance than would occur with the integrand of the denominator $\int_0^1 \frac{h_r}{K} \frac{\psi/\xi_e}{1+\beta P_r'} d(\psi/\xi_e)$ of formula (64).

192. Consequently with the aid of the relations (61) or (62) it may suffice to apply the "A" and "B" form for the function $\beta(\eta)$, provided the Pr-number is not too low.

193. The general expressions (formulas (61), (62) and (63)) for the dimensionless heat transfer coefficient ($h_{\theta u}'$) may be specialized for the conditions of the early theory. It implies Prandtl's dichotomy of the boundary layer in a purely laminar wall region ($\beta_{w} = 0$) and a purely turbulent outer region ($\beta_{outer} = \infty$). Thus, with the particular values $h_{r}=1$ i.e. $P_r' = P_r$, $I_K = 0$ the relations (61) and (62) of the advanced theory would become:

$$\left(h_{\theta u}'\right)_{early\ theory} = \frac{\varphi^{*2}}{1+(P_r-1)\varphi_a} \qquad (61')$$

$$\left(h_{\theta u^*}'\right)_{early\ theory} = \frac{\varphi^{*}}{1+(P_r-1)\varphi_a} \qquad (62')$$

194. In these formulas, $\varphi_a = u_a/U$ which denotes the relative velocity at the border between the two regions, is a purely hydrodynamical quantity. By comparing the particular formulas (primed) to the general relations (61) and (62) it follows that, in the actual third phase of the boundary layer theory for heat transfer, the counterpart of φ_a is the integral:

$$I_{i\ell}' = \varphi^* \int_0^{\psi^*} \frac{P_r'-1}{P_{rw}'-1} \frac{d\psi}{1+\beta P_r'} = \int_0^1 \frac{P_r'-1}{P_{rw}'-1} \frac{d\varphi}{1+\beta P_r'}$$

195. Even by simplifying with a constant ($P_{rw}'-1$), instead of including the variable ($P_r'-1$) in the integrand, still the simplified integral: $\varphi^* \int_0^{\psi^*} \frac{d\psi}{1+\beta P_r'} = \int_0^1 \frac{d\varphi}{1+\beta P_r'}$ will remain

a mixed thermo-aerodynamic quantity. This is due to its dependency in the denominator upon the generalized Prandtl-number (Pr') multiplied with the function $\beta = A_2/(\mu^+\eta)$. Physically, the integral $\int_0^1 d\varphi/1+\beta Pr'$ expresses the relative velocity for $\beta Pr' = A_2 c_p/k = 1$ i.e. at the relative distance from the wall for which $\eta = \eta_2$. There, heat motion by molecular agitation and by molar convection due to turbulence contribute equally to the heat transfer.

196. From the above analysis two methods of successive integration are evolved.

a. The heat transfer calculations consist in determining first by iterative steps of integration the distribution of both the temperature (function ϑ, formula (49) or (57)) and its gradient over the boundary layer. The wall value of the temperature gradient is then computed and entered in the relation for the heat transfer coefficient (formulas (58) to (63), References (o), (m), (r), (s)).

b. The heat transfer at the wall may be determined directly from the general relation for the temperature gradient at the wall (formula (56)) without seeking previously a full knowledge of an integrated thermal profile.

197. The first method is based on evaluating all of the integrals involved in the temperature profile. For a two step approximation their number is four. Two of them (I_1, I_2) are indefinite integrals, and two are definite (I_1 , I_2).

198. The second method implies the evaluation of the two definite integrals only. These are (formulas (62)) $I_{1\ell}$ and $I_{2\ell}$, and may be simplified by taking the factor $Pr'-1$ outside the integral as $Pr'_{av}-1$. The limits of integration are at both ends of the thermal boundary layer. Integrals with a variable upper limit are irrelevant here, although they should be indispensable for tracing the complete profile of the mean temperature from point to point along the normal to the wall.

199. As for the integral: $I_{1\ell} = \varepsilon = \int_0^1 \lambda\, d\varphi = \int_0^1 \left(\frac{q/q_0}{\tau/\tau_0} - 1\right) d\varphi$

the ratio $\varphi = \frac{q/q_0}{\tau/\tau_0}$ is strongly dependent upon the geometrical configuration of the wall.

200. For the flat plate: $\varphi \approx 1$ and $\varepsilon \approx 0$, for the open duct: ε though small, is not negligible, for the tube: $\varphi \neq 1$ since, at a closing wall, the gradients of shear stress and heat flow density are of opposite sign, and the relevant profiles of $\tau(\gamma)$ and $q(\gamma)$ may differ considerably.

Thus, the most important of the integrals involved is:

$$I'_{ie} = \varphi^{*} \int_{0}^{\Psi} \frac{P_r' - 1}{P_{r_{av}}' - 1} \frac{d\Psi}{1+\beta P_r'} = \int_{0}^{1} \frac{P_r' - 1}{P_{r_{av}} - 1} \frac{d\varphi}{1+\beta P_r'}$$

201. As the function $P_r' = (A_s/A_r)\, ^c p \mu / k$ depends on the poorly known ratio $m(\psi)$, it might as well be replaced for the time being by the average value $P_{r_{av}}$, and the discussion confined to the integral:

$$a = \int_{0}^{\Psi} \frac{d\Psi}{1+\beta P_r'} = \int_{0}^{\Psi} \frac{q_m}{q} d\Psi \qquad (65)$$

202. The form of the integrand is dependent upon the width (η_2) of the prevalently heat conducting layer near the wall. " η_2 " is that characteristic value of the variable for which the function $\beta(\eta)$ equals the reciprocal of the generalized Prandtl-number:

$$\beta(\eta_2) = 1/P_r' \qquad (66)$$

203. As was mentioned in Part II of this review (Reference (m_β)), the function β is independent of the Re-number as long as $\eta < 30$, and in the range: $6 < \eta < 30$, the form of β is:

$$\beta = K \left[\eta - \eta_m \tanh \sigma \right] \qquad (67)$$

where $\sigma = \eta/\eta_m$, while η_m is a measure of the width for the viscous boundary layer i.e. $\eta_m = f(\eta)_{\beta=1}$

204. For a region very close to the wall:

$$\lim_{\sigma \to \text{very small}} \tanh \sigma = \lim \left\{ \sigma - \left(\tfrac{1}{3}\right)\sigma^3 + \left(\tfrac{2}{15}\right)\sigma^5 - \cdots \right\} = \sigma - \sigma^3/3 \qquad (68)$$

while afar from the wall:

$$\lim_{\sigma \to \text{very large}} \tanh \sigma \approx 1$$

205. For small Pr-numbers, "η_2" expands into the region where the exchange of momentum is almost of a purely turbulent nature, and, with η_1 and $\sigma = \sqrt{\eta_n}$ tending to grow very large, so does the Re-number. Then:

$$\lim_{\sigma \to \text{very large}} \beta \approx \kappa(\eta - \eta_n) \tag{69}$$

206. Solving now the equation: $\kappa(\eta - \eta_n) = 1/P_r'$ yields:

$$\eta_2 = \eta_n(\eta_1) + \frac{1}{\kappa P_r'} \tag{70}$$

and the relative velocity function afar from the wall becomes:

$$\psi = \frac{\vartheta_m \eta}{k} + \psi_i \tag{71}$$

207. With the definition:

$$\eta_n = f(\eta)_{\beta=1} = f(\eta_1) \tag{72}$$

the validity of relation (70) is limited to regions with a high enough value of the Re-number.

208. For formulas (64):

$$St = \frac{h'_{v\theta}\,\theta}{u_m \mathcal{J}_m} = \frac{h'_{v\theta}}{\varphi_m \mathcal{J}_m} = \frac{Nu}{Re\ Pr} \tag{73}$$

and with $Re = 2\psi_m \eta_n$ it follows that:

$$St = \frac{Nu}{2\psi_m \eta_n Pr} \tag{74}$$

209. For a small Pr, the term $1/\kappa P_r'$ grows so large that, comparatively, the constant η_n may be neglected. Hence:

$$(\eta_2)_{\text{small } Pr} \approx 1/\kappa P_r'$$

210. Close to the wall, $m \approx 1$ i.e. $P_r' \approx P_r$. It follows that for small Pr and large Re:

$$P_r \approx 1/\kappa\eta_2$$

and:

$$P_r \eta_{r} \approx \frac{1}{\kappa(y_2/r)} \qquad (75)$$

211. It was found under the conditions envisaged in Reference (m), that the validity limit of the B-form (formula (67)) corresponds to $y/r \approx 0.5$. This figure is based on the best empirical data on turbulent flow in the tube. Hence, with $y_2/r = 0.5$ and $\kappa = 0.4$, the method of direct integration for the heat transfer coefficient at the wall would be limited to:

$$P_r \eta_r > \left(\frac{1}{\kappa\eta_2}\right)_{limit} = 50$$

212. It follows that, with $\eta_r = \frac{ru^*}{\nu} = \frac{Re}{2\varphi_m}$

$P_r Re > 100\,\varphi_m$ and with $\varphi_m \approx 25$ (Reference (m_J)):

$$P_r Re \geq 2500 \qquad (76)$$

Semiempirical Approach by Power Relations

213. Between differing conditions of a flow with heat transfer, the general criterion of similarity, as based on dimensional analysis, appears as a function "f" by which the heat transfer coefficient at the wall (in any form, say: Nu) is interrelated with six parameters, viz:

$$Nu = f\,(P_1, P_2, P_3, P_4, P_5, P_6) \qquad (77)$$

214. The arguments are:

$P_1 = Re = u_m d/\nu$: Reynolds number

$P_2 \equiv Pr' = m\frac{c_p\mu}{k}$: Generalized Prandtl-number

$P_3 = \mu_{fl}/\mu_w$: Viscosity ratio with subscript fl referring to the mean temperature of the fluid and w to the wall.

$P_4 = d/L$: Aspect ratio of the duct (open or closed)

$P_5 = T_e/T_{stagn.}$: Recovery ratio between the equilibrium temperature (T_e) at the wall and its stagnation temperature (T_{st}).

$P_6 = M = V/a$: Mach number

215. In time past it was observed that, ceteris paribus, the heat transfer with streaming gases, if compared to that with viscous liquids, appeared inversely affected by a reversal of the heat flow direction. Those experimental facts were rightly correlated to the opposition of sign as for temperature coefficients of viscosity between gases and liquids. Independency of the rate of heat transfer from its direction was secured by introducing a new argument (P_3) and thus correlating, ceteris paribus, Nu to a power of P_3 with an empirically found exponent.

216. The parameter $P_3 = \mu_{fl}/\mu_w$ accounts for a variable viscosity over the temperature profile. By neglecting P_3, the early investigations became affected with a dependency of the heat transfer coefficient at the wall upon the flow direction of the heat i.e. upon whether heat was moving toward the wall or away from it.

217. The parameter $P_5 = T_e/T_{stagn}$ may be implicitly introduced by taking the difference ($T_e - T_0$) between the equilibrium temperature (T_e) at the wall and its actual temperature (T_0) as reference in forming the heat transfer coefficient. Then:

$$Nu_e = f\left(Re,\ Pr',\ \mu_{fl}/\mu_w,\ d/L,\ M\right) \tag{78}$$

218. The Mach number extends the similarity requirements over the compared density fields provided the ratio c_p/c_v is linked to the Pr-number by a well defined relation as it occurs for the gases at least. For lower speed ranges the dependency on the Mach number may be neglected.

219. The semi-empirical procedure consists in giving to the unknown similarity function " f " the form of a product of powered individual parameters, and trying to determine empirically the unknown exponents. As a rule, power relations with a rather restricted range of validity are linked to a number of (1 + N) empirical constants with " N " denoting the number of independent similarity parameters (P_i) involved. Then Nu appears in the form:

$$Nu = c_o P_1^{c_1} P_2^{c_2} P_3^{c_3} \dots \tag{79}$$

220. In order to reduce the restrictions about the range of validity of such power relations it had been attempted to generalize by using binomial expressions of the form $c_k + P_i^{c_i}$ instead of $P_i^{c_i}$. Then the number of empirical constants in the generalized power relation exceeds that of the simple power formula by the number of factors in binomial form.

221. Of course increasing the number of empirical constants does not ease in general the work of determining them from a group of experimental series. Yet, for independency of the heat transfer from the heat flow direction, ceteris paribus, still the following generalized power relation (Reference (o)) would appear as one of the best available:

$$Nu = 0.116 \left(P_1^{2/3} - 125 \right) P_2^{1/3} P_3^{0.14} \left(1 + P_4^{2/3} \right) \tag{80}$$

222. The parameters P_1, P_2, P_4 are referred to the mean value between the wall temperature (T_w) and the mean temperature of the fluid (T_{fl}).

223. The coefficient 0.116 has been so chosen that, for an aspect ratio of D/L = 1/150, equation (80) is reduced to the form of a plane mean flow. Then:

$$Nu = \left(0.12 P_1^{2/3} - 15 \right) P_2^{1/3} P_3^{0.14} \tag{81}$$

224. A common weakness of all the semi-empirical heat transfer relations is the limited and approximate validity of such formulas within rather restricted ranges of the variables involved.

225. Reference (r) offers a juxtaposition with a discussion about ranges of validity for a great number of more or less empirical formulas of heat transfer computations in the form of a Nu-number or a Stanton-number: $St' = Nu / Pe'$

226. The same reference presents also, for turbulent heat transfer in the tube, a useful three-scale nomogram by which the St-number: $St = \left(\frac{d}{4L}\right) \frac{T_2 - T_1}{T_w - T_m}$ (82) is correlated to the Re-number and the Pr-number.

d	:	diameter
L	:	length
T_1	:	inlet temperature
T_2	:	outlet temperature
T_m	:	mean temperature
T_w	:	wall temperature

T_1, T_2, T_m: of fluid

227. In this nomogram the three variables cover respectively the following ranges:

$$(2.1)\,10^{2} \leq Re \leq (2.5)\,10^{4}$$
$$0 \leq Pr \leq 500$$
$$(3)\,10^{-5} \leq St \leq 1500$$

Comparison between Analytic Approach and Power Relations

228. To this end, the definite integral: $a = \int_0^{\Psi} \frac{q_m}{q}\, d\Psi$

has been evaluated (Reference (m$_5$)) over the range, $10^{-3} \leq Pr \leq 10^{3}$: with seven different values of the parameter Pr $(10^{-3}, 10^{-2}, 10^{-1}, 1, 10, 10^{2}, 10^{3})$. For each one, the function $q_m/q\,(\Psi)$ was plotted. The integrals were determined by measuring out the relevant areas.

a. Range of Median and Higher Pr-Number

229. In regions in which the integral "a" is independent of the Re-number, the relevant theoretical results were approxi-

mated (Reference (m)) by a power relation: $a_{theor.}$ of the Pr-number alone, according to the table below.

Total range of Pr	Pr-regions	Range of η	Theoretical Integral: $a_{theor.}$	Semi-empirical relation of ten Bosch
	median values	$6 \leq \eta < 30$	$(10)\ Pr^{-0.30}$	$a_{s.e.} = (10.08)\ Pr^{-0.145}$
$1 \leq Pr \leq 200$	high values	$\eta < 6$	$(9.12)\ Pr^{-0.20}$	

230. Thus, the theoretical power relations in the regions of validity of the B-form or the A-form of function ρ respectively (References (m) and (t)) appear in fair agreement with the corresponding semi-empirical formula with two constants as determined by Ten Bosch (Reference ($m_{\int}$)).

b. Range of Lower Pr- Numbers

231. As to measurements of turbulent heat transfer for very low Pr-numbers, the earlier experiments have been gravely handicapped because of the unsatisfactory wetting of the solid surface by such fluids as mercury flowing over current wall-materials. Those conditions yielded much too low heat transfer coefficients.

232. On the other hand, more recent measurements (Reference (s)) achieved notable progress by a more suitable choice of substances for the fluid and the wall respectively. Thus, the low melting alloy of sodium and potassium in equal parts ensured a full wetting over a wall of pure nickel.

233. In Reference (s), the comparison between experimental results and the theoretical formula

$$Nu = 7 + (0.025)\ Pe^{0.8} \tag{82}$$

revealed a satisfactory agreement for $Pe > 1000$. Below this upper limit, however, the measured values appeared as much as 30% under the computed ones.

234. Nevertheless, in Reference (q_p), the comparative discussion of the results of Reference (s) was renewed in the light of a generalized Prandtl-number. Thus with $Pe = Re \times Pr$, the formula for the turbulent heat transfer became:

$$Nu_{turb.} = 5.2 + (0.025)\, Pe^{0.8} \qquad (83)$$

235. It was shown there that this formula is well satisfied by both the theoretical results from iterative integration and the experimental ones within the range of the measurements (Reference (s)).

Part IV: Conclusions

236. The new theory of the boundary layer for turbulent flow (References (m) and (q)) is governed by:

a. The continuity of decay for both heat and momentum effects of turbulence from the free-stream to the wall.

b. A better compliance with the requirements of continuity for mass, momentum and energy in regard to both the mean flow and the turbulent motion. This led to the discovery of a very satisfactory form for the function: $\bar{p} = A_\tau / \mu$ over the entire boundary layer.

c. Interpreting the experiments of flow over heated bodies in the sense of assigning to the unknown function $m = A_q / A_\tau$, over the boundary layer, the border values $1 \leq m \leq 2$, and estimating $m_{aver.}$ for various flow conditions.

d. The use of $Pr' = mPr$ as one of the essential parameters in dealing with heat transfer problems of turbulent flow.

e. A distinction between two characteristic layer-widths (differing for $Pr' \neq 1$) which are: η_1 for viscous effects, η_2 for heat flux density. Each one marks a limit distance from the wall within which the relevant turbulent effect (τ_t or q_t) does not exceed its molecular counterpart (τ_m or q_m).

237. Based on these five points, the calculations of the temperature distribution and heat transfer have been extended over a range of:

a. High values of the Pr-number (References (f) and (m)) up to $Pr' = 10^3$.

b. Low Pr-numbers (References (q), (m), and (s)) down to $Pr' = 10^{-3}, 0$).

238. The former category corresponds to nonmetallic liquids like water and oils. The latter covers conditions of metals in liquid state (mercury, molten metals and alloys). The lower range became of particular importance because of the necessity of removing heat in large amounts from nuclear furnaces, at a high temperature level, by applying molten metals as coolants.

239. The shape of the temperature profile appears to be highly dependent upon the generalized Pr-numbe. (Pr' = mPr), whereas it seems less affected by the Re-number. However, the dimensionless heat flux density (q/q_0) is found more dependent on the Re-number.

240. Almost irrespective of both parameters (Pr' and Re) the average mean temperature for the turbulent flow in a circular tube appeared at a distance from the wall equal to 0.3 of the tube radius (References (p) and (q)).

241. As for the task laying ahead of us in this field, let the following be hinted. A considerable need seems to subsist for exploring still more by systematic experiments the profiles of both velocity and temperature for various conditions of turbulent flow. Increased accuracy of results might allow one to reach by inference the form of the characteristic function: m = $\bar{A}_q / \bar{A}_\tau$. This would be an important contribution toward a better understanding of what appears to be a composite structure of turbulence.

REFERENCES *

(a) Reynolds, O.,

"On the Extent and Action of the Heating Surface of Steam Boilers." Proc. Manchester Lit. and Phil. Soc. Vol. 14, 1874, pp. 7-12. Also in Collected Papers, Cambridge, England Vol. 1, No. 14, 1900, pp. 81-85.

(b) Stanton, T. E.,

(α) "On the Passage of Heat between Metal Surfaces and Liquids in Contact with them." Phil. Trans. Roy. Soc. A 190, 1897, pp. 67-88.

(β) "The Relation between Friction and Heat Transmission in the Case of the Motion of a Fluid over a Surface having Temperature differing from that of the Fluid." Dict. Appl. Physics. Vol. 1, Art. Friction. VI § 40-41, 1922, pp. 399-404.

(c) Prandtl, L.,

(α) "A Relation between Heat Exchange and Drag" "Eine Beziehung zwischen Wärmeaustausch und Strömungswiderstand der Flüssigkeiten." Phys. Zeitschr. 11, 1910, pp. 1072-1078.

(β) "Remarks on the Heat Transfer in the Tube." "Bemerkungen uber den Wärmeübergang im Rohr." Phys. Z. 29, 1928, pp. 487-489.

(d) Taylor, G. I.,

(α) "Eddy Motion in the Atmosphere." Phil. Trans. Roy. Soc. A 215, 1915, pp. 1-33.

(β) "The Transport of Vorticity and Heat through Fluids in Turbulent Motion." Proc. Roy. Soc. A 135, 1932, pp. 685-705.

---

* All reference titles are given in English. However, for a publication in another language, the original title follows its translation.

REFERENCES (CONT'D)

(e) Stodola, A.,

"To the Theory of Heat Transfer from Gases or Liquids to Solid Walls" -- "Zur Theorie des Wärmeüberganges von Gasen oder Flüssigkeiten an feste Wände." Schweiz. Bauztg. 1926, pp. 243-244, 1927, pp. 193-196.

(f) von Kármán, Th.,

(a) "Turbulence and Skin Friction." Journ. IAS 1934, pp. 1-20.

(b) "The Analogy between Fluid Friction and Heat Transfer" ASME, Nov. 1939, pp. 705-710.

(g) Elias, F.,

"Heat Transfer at a Heated Plate to Flowing Air" -- "Die Wärmeübertragung einer geheizten Platte an strömende Luft." Abhandl. Aachen, Heft 9, 1930. Also ZAMM, 1929, pp. 434-453, II 1930, pp. 1-14.

(h) Colburn, A. P.,

"A Method of Correlating Forced Convection Heat Transfer Data, and a Comparison with Fluid Friction." Trans. Am. Inst. Chem. Eng. Vol XXIX, 1933, pp. 174-210.

(i) Lorenz, H., and Friedrichs,

"Contribution to the Problem of Heat Transfer" "Beitrag zum Problem des Wärmeübergangges." Zeitschr. techn. Physik, 1934, pp. 156-162, 201-206, 376-377.

(j) Kaye, W. A., and Furnas, C. C.,

"Heat Transfer involving Turbulent Fluids", Industr. Engin. Chem. Vol. 26, 1934, pp. 783-786.

REFERENCES (CONT'D)

(k) Sieder, E. N., and Tate, G. E.,

"Heat Transfer and Pressure Drop of Liquids in Tubes", Industr. Eng. Chem. Vol. 28, 1936, pp. 1429-1435.

(l) Eckert, E.,

"The Heat Transfer for Cooling and Heating" -- "Der Wärmeübergang beim Kühlen and Heizen." Zeitschr. VDI, 1936 pp. 137-138.

(m) Reichardt, H.,

(α) "Heat Transfer through Turbulent Friction Layers." NACA, TM 1047, Dec. 1943, 58 pages, translation from ZAMM, Dec. 1940, pp. 297-328.

(β) "The Influence of the Flow Near the Wall Upon the Turbulent Heat Transfer" -- "Der Einfluss der wandnahen Strömung auf den turbulenten Wärmeübergang." Mitteil. Nr. 3 of Max Planck Inst., Göttingen 1950, 63 pages.

(γ) "Complete Representation of the Turbulent Distribution of Velocity in Smooth Ducts" -- "Vollständige Darstellung der turbulenten Geschwindigkeitsverteilung in glatten Leitungen." ZAMM, July 1951, pp. 208-218.

(δ) "The Foundations of the Turbulent Heat Transfer" -- "Die Grundlagen des turbulenten Wärmeüberganges." Arch. ges. Wärmetechnik, July - August 1951, pp. 129-142.

(n) von Mises, R.,

"Some Remarks on the Laws of Turbulent Motion in Tubes." Von Kármán Anniversary Volume, 1941, pp. 317-327.

(o) Hausen, H.,

"Representation of the Heat Transfer in Tubes by Generalized Power Relations" - "Darstellung des

REFERENCES (CONT'D)

Wärmeübergenges in Rohren durch verallgemeinerte Potenzbeziehungen." VDI Verfahrenstechnik, 1943, Nr. 4, pp. 91-98.

(p) Jakob, M.,

Discussion by,

(α) Trans. Am. Inst. Chem. Eng. 1940, pp. 657-662.

(β) Trans. ASME, 69, 1947, p. 958.

(q) Elser, K.,

(α) "Temperature Fields due to Viscosity in Turbulent Boundary Layers" -- "Reibungstemperaturfelder in turbulenten Grenzschichten." Mitteil, Nr. 8, Inst. Thermodyn. and Design Comb. Motors, ETH. Zürich, 1949, 88 pages.

(β) "The Turbulent Heat Transfer in the Tube at Very Small Prandtl Numbers" -- "Der turbulente Wärmeübergang im Rohr bei sehr kleinen Prandtl-Zahlen." Arch. ges. Wärmetechn. Nr. 10, 1951, pp. 206-211.

(r) Koch, R.,

"Turbulent Heat Interchange in the Tube" -- "Turbulenter Wärmeaustausch im Rohr." Arch. ges. Wärmetechn., Jan. 1950, pp. 2-8.

(s) Lyon, R. N.,

"Liquid Metal Heat Transfer Coefficients." Chem. Eng. Progress, February 1951, pp. 75-79.

(t) Theodorides, Ph. J.,

"Turbulence and Heat Transfer at Smooth Walls for a Flow in a Prismatic Duct or along a Flat Plate" (Unpublished Survey of Some Recent Developments), 1952, 72 pages

ILLUSTRATIONS

ILLUSTRATIONS (CONT'D)

Analogous Coefficients (b_i) for Time Rate of Molecular, and of Turbulent Transport of Momentum, Heat, and Mass, respectively, per Unit area across the Mean Flow Direction

$$\boxed{\sigma_i = b_i \frac{\partial \bar{\varepsilon}}{\partial y}}$$

Time Rate of Transport per Unit Area			Coefficient of Physical Property				
σ_i	Molecular	Turbulent	b_i	Molecular	Turbulent	Temporal mean value $\bar{\varepsilon}$	Gradient
$\frac{\text{Momentum}}{(\text{time})\ (\text{area})}$	τ_m	τ_t	viscosity	μ	A_τ	velocity: $\bar{u}$	$\frac{\partial \bar{u}}{\partial y}$
$\frac{\text{Enthalpy}}{(\text{time})\ (\text{area})}$	q_m	q_t	heat conductivity (k) specific heat (c_p)	$-k/c_p$	$-A_q$	enthalpy: $\bar{i}$ $\bar{i} = \int c_p\, d\bar{\theta}$ temp.: $\bar{\theta}$	$\frac{\partial \bar{i}}{\partial y} = c_p \frac{\partial \bar{\theta}}{\partial y}$
$\frac{\text{Mass}}{(\text{time})\ (\text{area})}$	r_m	r_t	mass diffusion	$-\chi$	$-A_r$	concentration: $\bar{c}$	$\frac{\partial \bar{c}}{\partial y}$
Example: Total heat flow per unit cross-area: $q = -\left(\frac{k}{c_p} + A_q\right)\frac{\partial \bar{i}}{\partial y} = -\left(k + c_p A_q\right)\frac{\partial \bar{\theta}}{\partial y}$			All coefficients depend on the temperature; moreover the turbulent parts vary also with position of whirling element.				

Favored attitude for axes of eddies	Main flow	A_q/A_τ
Normal to direction of main flow	free jet	~ 2
Parallel to direction of main flow	about thin flat plate	~ 1.4 (to ~ 1.5)
	in tube	~ 1.1 (to [illegible])

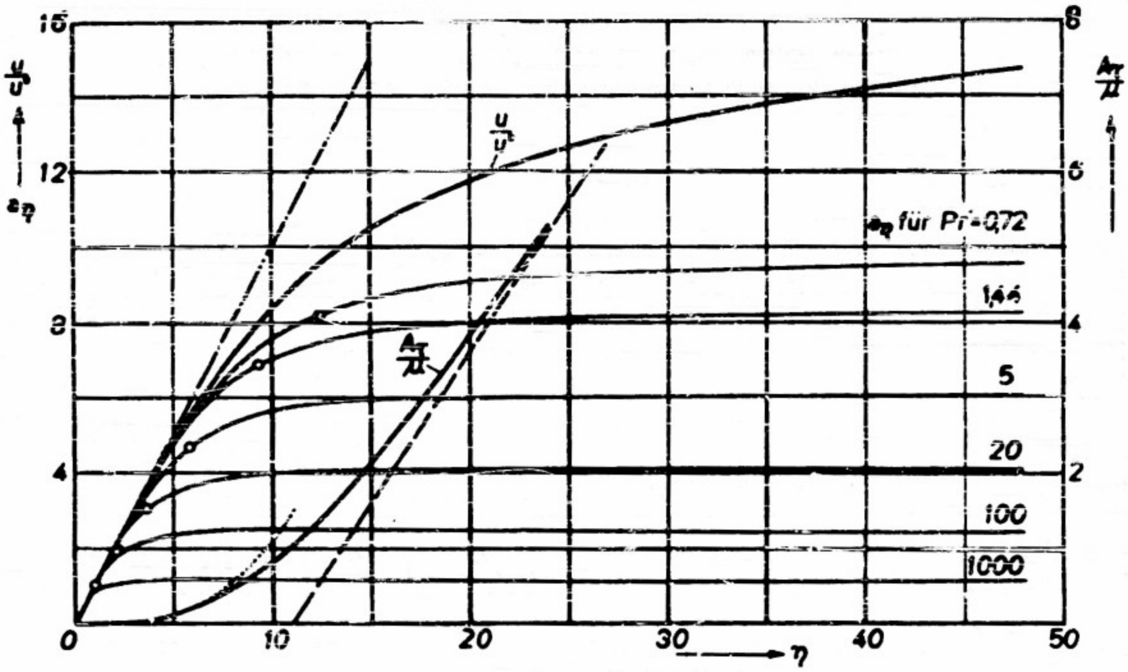
Symbols for Accepted Essentials in Presenting a Fluctuating Physical Quantity

	Additive fluctuation above mean value	Global x-component of velocity
Instantaneous	u_f	$u = \bar{u} + u_f$
Temporal mean	$\bar{u}_f = \frac{1}{T}\int_0^T u_f\, dt = 0$	$\bar{u} = \bar{u} + 0$
Effective	$u_f' = \left(\overline{u_f^2}\right)^{\frac{1}{2}} = \left(\frac{1}{T}\int_0^T u_f^2\, dt\right)^{\frac{1}{2}}$	$U = \bar{u} + u_f'$

The temporal mean square value $\left(\overline{u_f^2}\right)$ or the square root of it (r.m.s. – value): $\left(\overline{u_f^2}\right)^{\frac{1}{2}}$ are accepted measures of turbulence intensity

Figure 1

Figure 1 is to illustrate the accepted ways of expressing the effects of turbulent agitation on the transport of momentum, enthalpy and mass. It is in close analogy to the formalism inspired by Newton and Fourier for viscous and heat conducting effects of thermal agitation. In addition, Figure 1 includes the usual breakdown of fluctuating physical quantities.

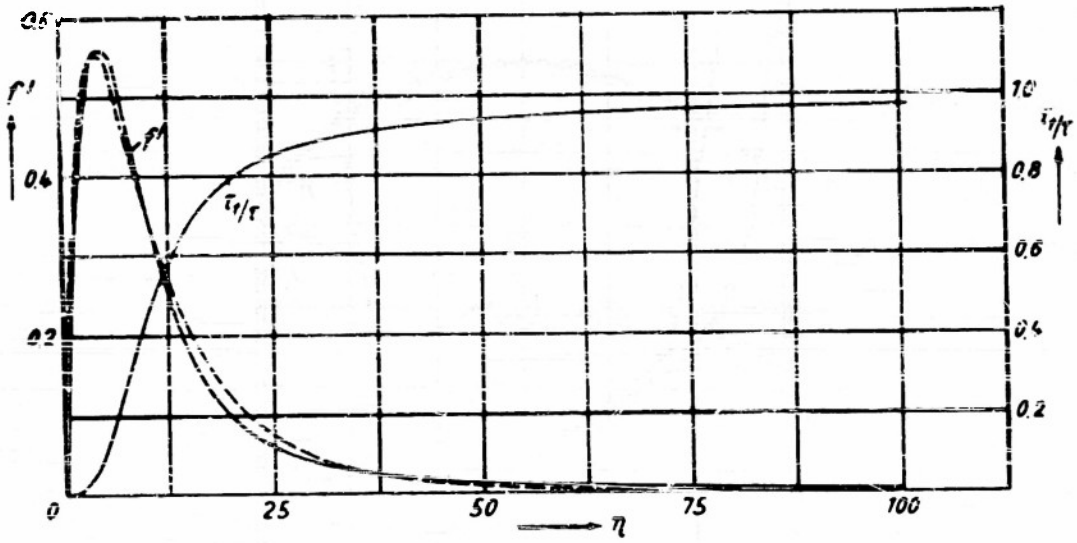


Ratio (A_τ/μ) of turbulent to viscous friction, relative velocity (u/u^*), and quantity a_η (at different Prandtl numbers) in terms of the non-dimensional distance ($\eta = yu^*/\nu$) from the wall.

(1)

(Graph reproduced from: H. Reichardt, Mitteilung No. 3 of Max Planck Institute, Göttingen, 1950, Abb. 1)

Figure 2



$$\int_0^{\eta} f'(\eta)\, d\eta =$$

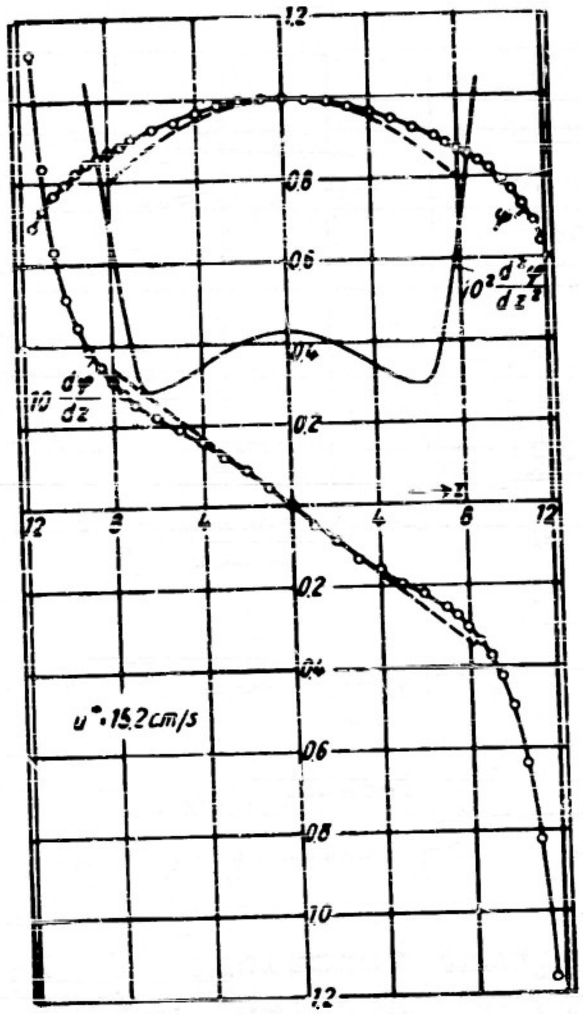
$$\int_0^{\eta} \frac{\kappa\eta_1 \tanh \eta/\eta_1 \, d\eta}{(1+\kappa\eta-\kappa\eta_1 \tanh \eta/\eta_1)(1+\kappa\eta)}$$

The integrand function: f' (η), and the ratio of turbulent part to total shear stress over non-dimensional distance (η) from wall.

(2)

(Graph reproduced from: H. Reichardt, ZAMM, July 1951, Bild 2)

Figure 3

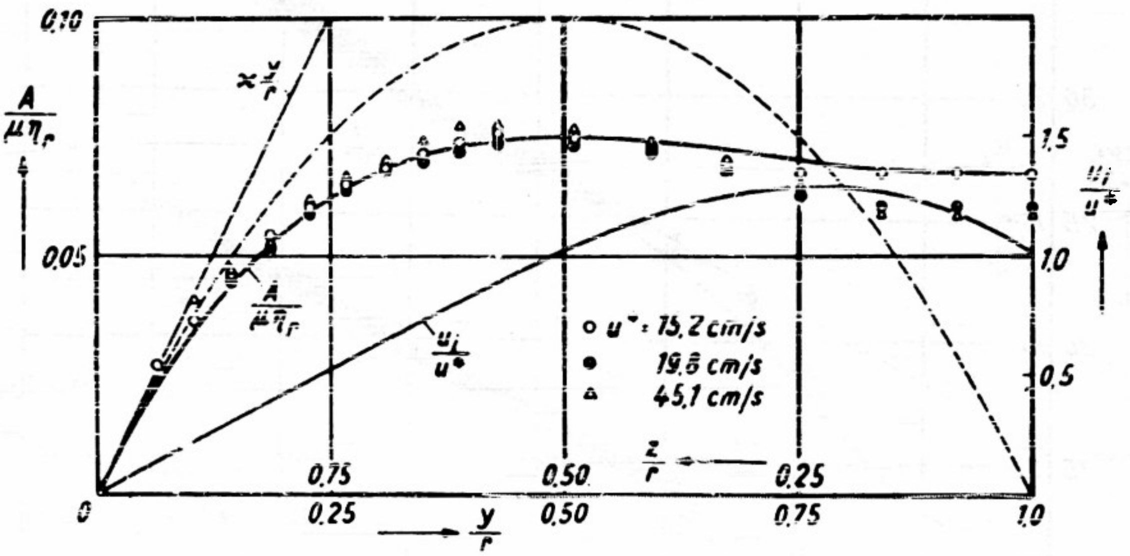


Relative velocity (φ) and cross-derivatives thereof ($d\varphi/dz$, $d^2\varphi/dz^2$) over a rectangular channel.

(3)

(Graph reproduced from: H. Reichardt, ZAMM, July 1951, Bild 4)

Figure 4

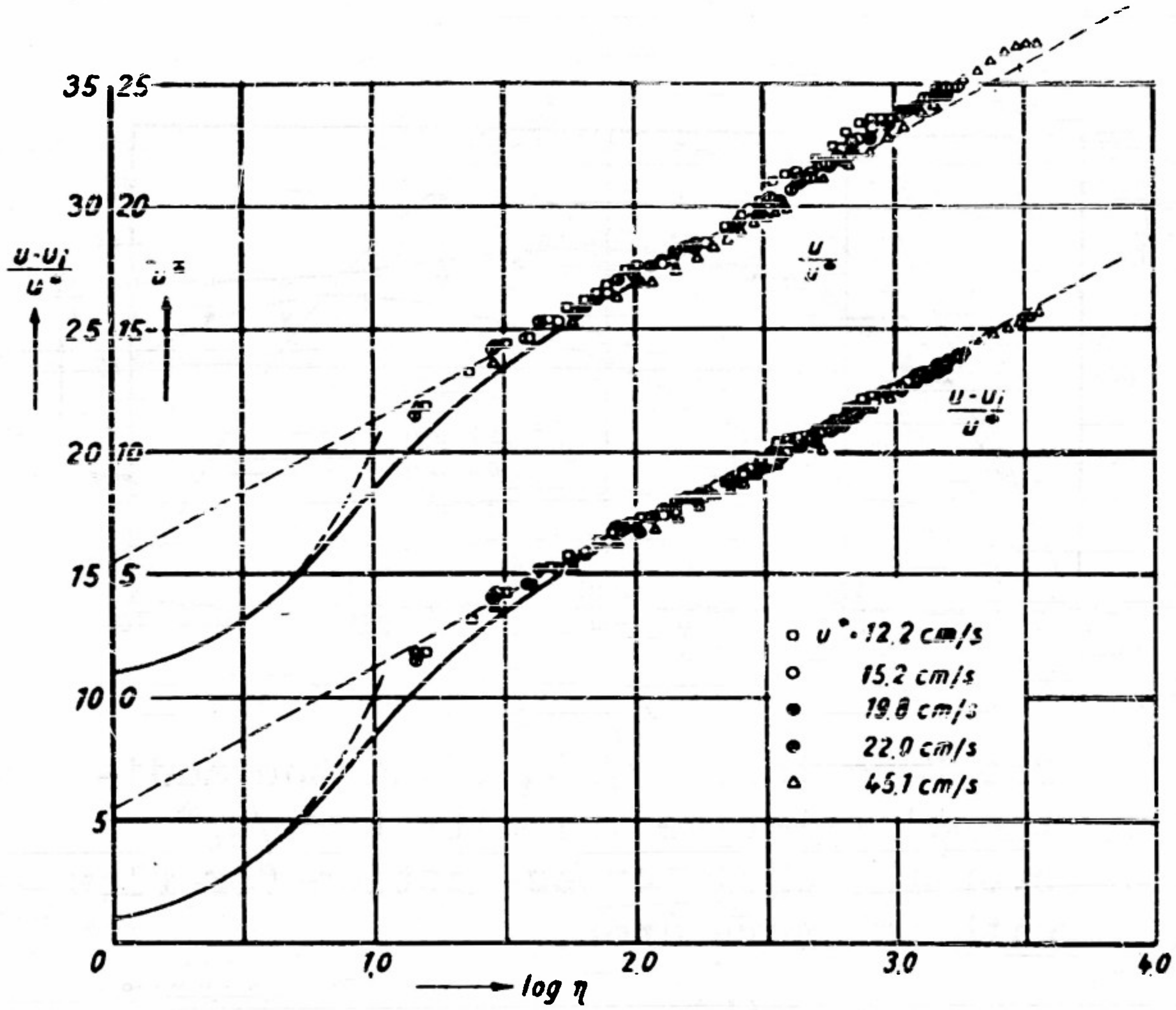


The function: $A/\mu\cdot\eta_r$, and the additional relative velocity (u_i/u^*) over the whole cross-section for flow with pressure drop.

(4)

(Graph reproduced from: H. Reichardt, ZAMM, July 1951, Bild 5)

Figure 5

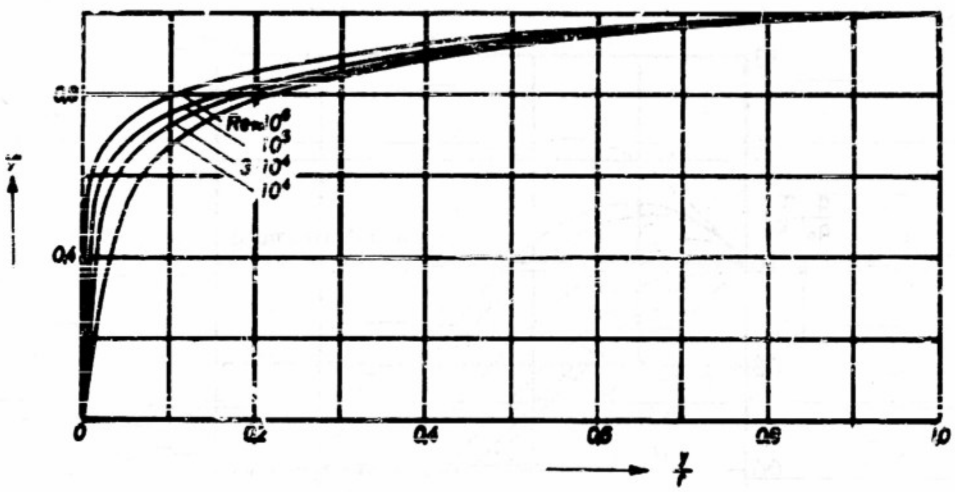


Comparative plots of relative velocity with and without corrective term.

(5)

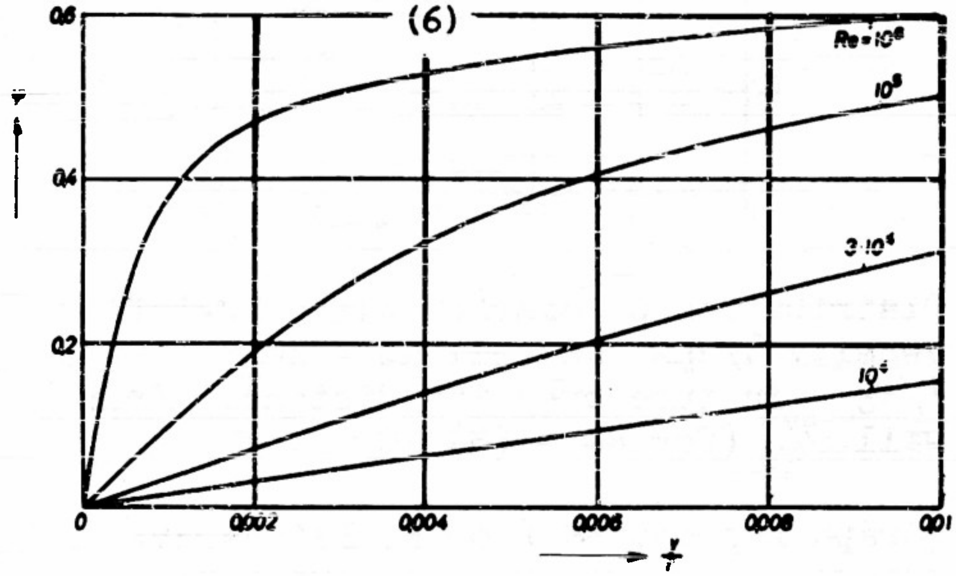
(Graph reproduced from: H. Reichardt, ZAMM, July 1951, Bild 7)

Figure 6



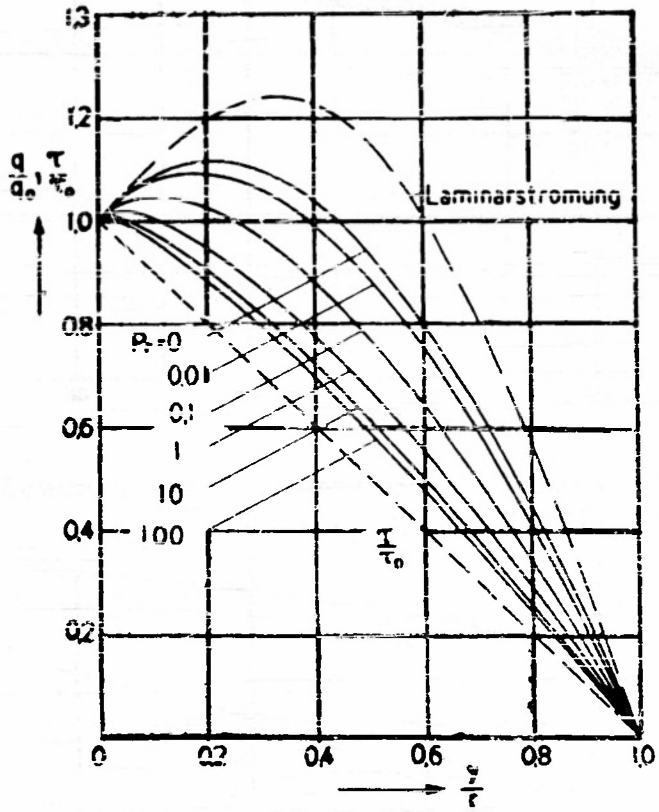
Complete velocity profile of flow in a cylindrical tube for different Re-numbers.

(6)



(Graphs reproduced from : H. Reichardt, Mitteilung No. 3 of Max Planck Institute, Göttingen, 1950, Abb. 3 and Abb. 4)

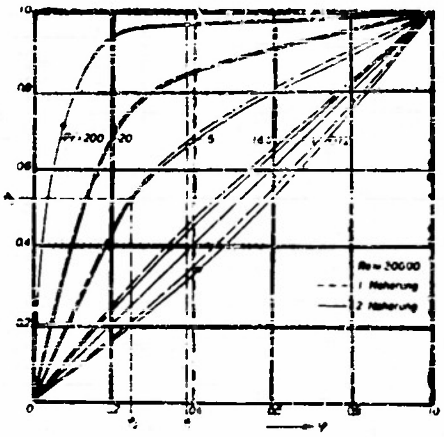
Figure 7



Distribution of dimensionless heat flux density q/q_o as well as shear stress τ/τ_o in tube over the distance from wall y/r (for Re = (3) 10^4)

(Graph reproduced from H. Reichardt: Grundlagen des turbulenten Wärmeüberganges, Arch. ges. Wärmetechn. 1951, H. 6/7, Fig. 5, p. 136)

Figure 8



Temperature profiles over non-dimensional velocity (φ) and over non-dimensional distance from wall.

(7)

(Graphs reproduced from: H. Reichardt, Mitteilung No. 3 of Max Planck Institute, Gottingen, 1950, Abb. 6 and Abb. 7)

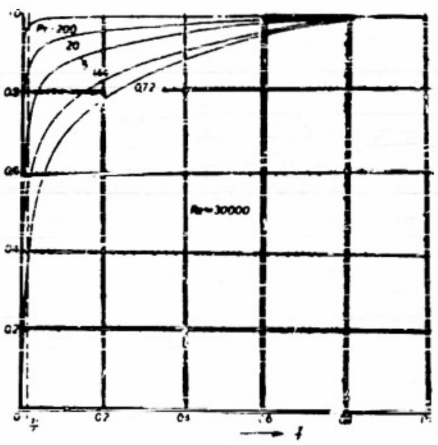
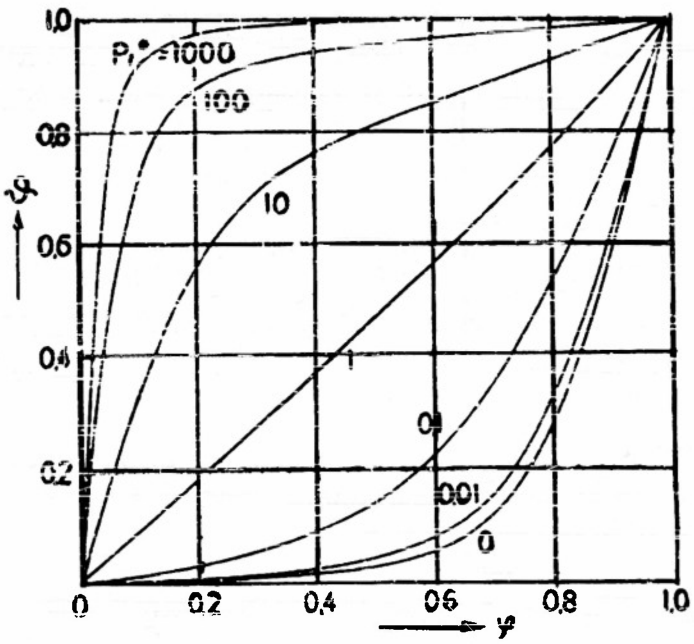


Figure 9

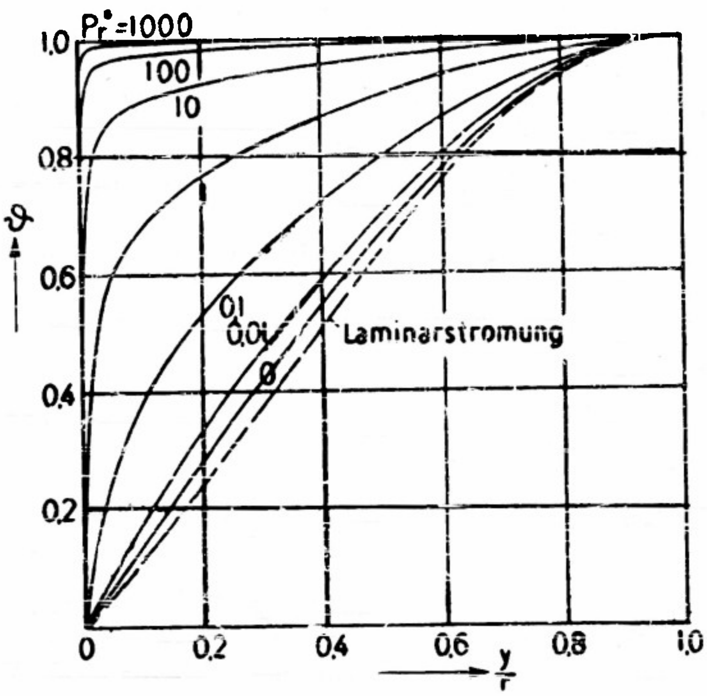


Temperature distribution (ϑ) versus velocity (φ).

Turbulent flow in tube (second approximation) for Re=(3)10^4 ($\eta_R = 800$) at various Prandtl-numbers.

(Graph reproduced from H. Reichardt: Grundlagen des turbulenten Wärmeüberganges, Arch. ges. Wärmetechn. 1951, H. 6/7, Fig. 3, p. 135)

Figure 10



Temperature distribution (ϑ) versus dimensionless distance from wall (y/r).

Turbulent flow in tube (second approximation) for Re=(3)10^4 at various Prandtl-numbers.

(Graph reproduced from H. Reichardt: Grundlagen des turbulenten Wärmeübergangs, Arch. ges. Wärmetechn. 1951, H. 6/7, Fig 4, p. 136)

Figure 11

No. of
Copies

Office of Naval Research
Department of the Navy
Washington 25, D. C.
1 Attn: Code 438
2 Attn: Code 463

Director
Naval Research Laboratory
Washington 25, D. C.
1 Attn: Code 2021

Commanding General
Wright Air Development Center
Wright-Patterson Air Force Base
Dayton, Ohio
1 Attn: WCAPD

Commanding General
Aberdeen Proving Ground
Aberdeen, Maryland
1 Attn: C. L. Poor

National Bureau of Standards
Corona Laboratories (Ord. Dev. Div.)
Corona, California
1 Attn: Dr. H. Thomas

National Bureau of Standards
Building 3U, UCLA Campus
405 Hilgard Avenue
Los Angeles 24, California
1 Attn: Librarian

University of California
211 Mechanics Building
Berkeley 4, California
1 Attn: Dr. S. A. Schaaf
VIA: InsMat

California Institute of Technology
Pasadena 4, California
2 Attn: Librarian (Guggenheim Aero Lab)
1 Attn: Dr. H. T. Nagamatsu
VIA: BuAero Representative

No. of
Copies

Massachusetts Inst. of Technology
Cambridge 39, Massachusetts
1 Attn: Prof. G. Stever
1 Attn: Prof. J. Kaye
VIA: InsMat

University of Michigan
Ann Arbor, Michigan
1 Attn: Prof. Otto Laporte
VIA: InsMat

University of Minnesota
Rosemount, Minnesota
1 Attn: Prof. N. Hall
VIA: Ass't InsMat

The Ohio State University
Columbus, Ohio
1 Attn: G. L. Von Eschen
VIA: Ass't InsMat

Polytechnic Institute of Brooklyn
99 Livingston Street
Brooklyn 2, New York
1 Attn: Dr. Antonio Ferri
VIA: ONR

Princeton University
Princeton, New Jersey
1 Attn: Prof. S. Bogdonoff
VIA: ONR

Applied Physics Laboratory
The Johns Hopkins University
8621 Georgia Avenue
Silver Spring, Maryland
1 Attn: Arthur G. Norris
VIA: NIO

1 Cornell Aeronautical Lab., Inc.
4455 Genesee Street
Buffalo 21, New York
VIA: BuAero Rep.

External Distribution List for NavOrd Report 2958

No. of Copies	
	Harvard University Pierce Hall Cambridge 38, Massachusetts
1	Attn: Librarian, Gordon McKay Engineering Library
	University of Maryland College Park, Maryland
1	Attn: Professor M. H. Martin
1	Attn: Dr. Th. Theodorsen, Consultant, USAF
5	Attn: Professor Ph. J. Theodorides
	Massachusetts Institute of Technology Cambridge 39, Massachusetts
1	Attn: Librarian, Guggenheim Aero-Library
	Applied Physics Laboratory The Johns Hopkins University 8621 Georgia Avenue Silver Spring, Maryland
1	Attn: Dr. F. N. Frenkiel
	National Advisory Committee for Aeronautics 1724 F Street, N. W. Washington 25, D. C.
1	Attn: Dr. H. Dryden, Director
	Max Planck Institut fur Strömungsforschung Göttingen - Böttingerstr. 6 West Germany
1	Attn: Dr. H. Reichardt
	Institute for the Aeronautical Sciences, Inc.
1	Attn: Librarian, 2 East 64th Street, New York 21, N. Y.
1	Attn: Librarian, Western Headquarters, Los Angeles, Cal.
	Royal Aeronautical Society 4 Hamilton Place W1, London, England
1	Attn: Librarian

No. of Copies	
	Advisory Group for Aero-R&D of NATO AGARD, APO 230 c/o Postmaster, New York, N. Y.
1	Attn: Professor Th. von Karman, Chairman
1	Attn: Dr. Frank Wattendorf, Director
	Eidg. Technische Hochschule Zürich, Switzerland
1	Attn: Librarian, Central Building
	Chief, Bureau of Aeronautics Department of the Navy Washington 25, D. C.
1	Attn: AER-TD-414
1	Attn: RS-7
	Commander U. S. Naval Ordnance Test Station Inyokern P. O. China Lake, California
1	Attn: Technical Library
	Commander U. S. Naval Air Missile Test Center Point Mugu, California
2	Attn: Technical Library
	Superintendent U. S. Naval Postgraduate School Monterey, California
1	Attn: Librarian
	Commanding Officer and Director David Taylor Model Basin Washington 7, D. C.
1	Attn: Hydrodynamics Laboratory
	Chief of Naval Research Library of Congress Washington 25, D. C.
1	Attn: Technical Info. Div.

No. of Copies	
1	Defense Research Laboratory University of Texas Box 1, University Station Austin, Texas VIA: InsMat
	Consolidated Vultee Aircraft Corp. Daingerfield, Texas
1	Attn: J. E. Arnold VIA: Dev. Contract Office
	North American Aviation, Inc. 12214 Lakewood Boulevard Downey, California
1	Attn: Aerophysics Library VIA: BuAero Representative
	National Advisory Committee for Aero 1724 F Street, Northwest Washington 25, D. C.
1	Attn: E. B. Jackson
	Ames Aeronautical Laboratory Moffett Field, California
1	Attn: H. J. Allen
	NACA Lewis Flight Propulsion Lab. Cleveland Hopkins Airport Cleveland 11, Ohio
1	Attn: John C. Evvard
	Langley Aeronautical Laboratory Langley Field, Virginia
1	Attn: Dr. Adolf Buseman
1	Attn: Mr. J. Stack
	Harvard University 21 Vanserg Building Cambridge 38, Massachusetts
1	Attn: Prof. Garrett Birkhoff
	The Johns Hopkins University Charles and 34th Streets Baltimore 18, Maryland
1	Attn: Dr. Francis H. Clauser

No. of Copies	
1	Dr. Allen E. Puckett, Head Missile Aerodynamics Department Hughes Aircraft Company Culver City, California
1	Dr. Gordon N. Patterson, Director Institute of Aerophysics University of Toronto Toronto 5, Ontario, Canada VIA: BuOrd (Ad8)
1	Office of Naval Research Branch Office Navy 100 Fleet Post Office New York, New York
	Commanding General Aberdeen Proving Ground Aberdeen, Maryland
1	Attn: Dr. B. L. Hicks
	National Bureau of Standards Aerodynamics Section Washington 25, D. C.
1	Attn: Dr. G. B. Schubauer, Chief
	University of Minnesota Dept. of Aeronautical Engr. Minneapolis, Minnesota
1	Attn: Professor R. Hermann VIA: Ass't InsMat
	Case Institute of Technology Dept. of Mechanical Engineering Cleveland, Ohio
1	Attn: Professor G. Kuerti VIA: ONR